MASTERARBEIT / MASTER THESIS

Titel der Masterarbeit / Title of the Master Thesis

"Cyclic Cohomology of Non-commutative Tori"

verfasst von / submitted by

Mischa Matthias Elkner, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2022 / Vienna, 2022

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Shantanu Dave, Privatdoz. PhD

Abstract (English)

This thesis is about non-commutative tori, in particular their cyclic cohomology. We explicitly compute the (periodic) cyclic cohomology of the pre-C* algebra of smooth functions on a non-commutative torus, along the lines of Nest [13]. The calculation consists of two steps: in the first, we describe the Hochschild cohomology using a projective resolution; the second step employs a spectral sequence which relates the Hochschild cohomology with the periodic cyclic cohomology. As an application, we discuss the index theory of certain elliptic difference-differential equations associated to non-commutative tori, as developed by Connes [6]. Preliminary facts about non-commutative geometry are summarised in the first chapter. This work is expository rather than original, the aim being to give a detailed account of the previously mentioned results.

Abstract (Deutsch)

Thema dieser Arbeit sind nichtkommutative Tori, insbesondere deren zyklische Kohomologie. Für die Prä-C*-Algebra der glatten Funktionen auf einem nicht-kommutativen Torus wird, basierend auf der Arbeit von Nest [13], die (periodische) zyklische Kohomologie explizit berechnet. Die Berechnung besteht aus zwei Teilschritten: Im ersten Schritt beschreiben wir die Hochschild-Kohomologie über eine projektive Auflösung; im zweiten Schritt wird über eine Spektralsequenz auf die periodische zyklische Kohomologie geschlossen. Als Anwendung wird die von Connes [6] entwickelte Indextheorie gewisser elliptischer Differenz-Differentialgleichungen, die mit nichtkommutativen Tori assoziiert sind, diskutiert. Die notwendigen Grundlagen aus nichtkommutativer Geometrie sind im ersten Kapitel zusammengefasst. Ziel dieser Arbeit ist eine detaillierte Darstellung dieser bereits bekannten Resultate.

Acknowledgements

First and foremost, I would like to thank my advisor Shantanu Dave for his guidance, his valuable advice (not only on mathematics), his patience in working with me, and for introducing me to magical parts of mathematics I had not seen before.

During my master's studies, I have received help and support from more people than I can list here. Still, I would like to especially thank Andreas Čap, Nils Carqueville, Willi Kepplinger and Vera Vértesi, who were there for me whenever I needed their help.

Also always there for me are my partner and my family. Their support, advice and love are a great source of energy for me, and I am truly grateful to have them.

I thank my partner Mirjam Bissmeier for proofreading this thesis.

Contents

1 Preliminaries

This chapter serves as a brief introduction to C*-algebras, the basic idea of non-commutative geometry, cyclic cohomology and its relation to (noncommutative analogues of) vector bundles. In particular, we discuss how commutative C*-algebras are dual to (well-behaved) topological spaces. This motivates thinking of general C*-algebras as generalised, "non-commutative" topological spaces. Afterwards, we give a minimal introduction to cyclic cohomology, which can be viewed as a suitable generalisation of de Rham (co)homology. In particular, there is a generalised version of the relationship between complex vector bundles over a manifold and its cohomology, this being the topic of section 1.4. References for this chapter are [1] (particularly for the first section), [9], and [6] (which is closely followed in sections 1.3 and 1.4).

1.1 C*-algebras

1.1.1 First definitions

A locally compact Hausdorff space X can be studied via the space $C(X)$ of continuous functions $X \to \mathbb{C}$. Pointwise multiplication and addition turn $C(X)$ into a $\mathbb{C}$-algebra, and with the supremum norm it becomes a Banach algebra. There is one more important operation, namely the involution $C(X) \to C(X)$ given by pointwise complex conjugation. The resulting structure is an example of a *-algebra:

Definition 1.1. A *Banach *-algebra* is a Banach algebra A together with an isometric involutive antilinear antiautomorphism $A \to A, x \mapsto x^*$. That is, we have

(i) $\|x^*\| = \|x\|$,

(ii) $(x^*)^* = x$,

(iii) $(x + \lambda y)^* = x^* + \overline{\lambda} y^*$,

(iv) $(xy)^* = y^* x^*$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

Another important example of a *-algebra is provided by $\mathcal{L}(H)$, the bounded linear operators on a Hilbert space H, with taking adjoints as the involution. For both $C(X)$ and $\mathcal{L}(H)$, the norm and the involution in fact fulfil a stronger identity than (i), the so-called *C*-property*:

Definition 1.2. A *C*-algebra* is a Banach *-algebra A which fulfils the *C*-property*:

$$\forall x \in A : \quad \|x^*x\| = \|x\|^2.$$

To simplify notation, we will usually denote all *-involutions (as well as all norms) with the same symbols.

For $C(X)$ the C*-property directly follows from the fact that $\overline{z}z = |z|^2$ for $z \in \mathbb{C}$. For a bounded linear operator $T \in \mathcal{L}(H)$, the computation

$$\begin{aligned}\|T\|^2 = (\sup_{\|v\|=1} \|Tv\|)^2 = \sup_{\|v\|=1} \|Tv\|^2 = \sup_{\|v\|=1} \langle T^*Tv, v\rangle \leq \sup_{\|v\|=1} \|T^*Tv\|\|v\| \leq \\ \leq \|T^*T\|\end{aligned}$$

shows $\|T^*T\| \geq \|T\|^2$, and the reverse inequality is clear from submultiplicativity of the operator norm.

The C*-property is indeed stronger than isometry. On the one hand, any element x of a C*-algebra A fulfils

$$\|x\|^2 = \|x^*x\| \leq \|x\|\|x^*\|$$

and thus $\|x\| \leq \|x^*\|$, and upon replacing x by x^* we conclude $\|x\| = \|x^*\|$. On the other hand, it is not hard to construct Banach *-algebras without the C*-property. In fact, the norm on a C*-algebra A is completely determined by the remaining (algebraic) structure due to the identity

$$\forall x \in A : \quad \|x\| = \|x^*x\|^{1/2} = r(x^*x),$$

where $r(x^*x)$ is the spectral radius of x^*x. Consequently, for a given algebra with involution $*$ as above, there is at most one, but possibly no norm which makes it a C*-algebra. This indicates that C*-algebras are considerably more structurally rigid than general Banach *-algebras.

We next define the appropriate notions of morphisms between C*-algebras, and of C*-subalgebras.

Definition 1.3. Let A, B be C*-algebras. A *morphism* from A to B is an algebra homomorphism $\phi\colon A \to B$ such that $\phi(a^*) = \phi(a)^*$, for all $a \in A$.

Of course, a morphism should also be continuous, but this follows from the definition. In fact, one can show that every morphism of C*-algebras is norm-decreasing [1, p. 29]. In particular, if a morphism ϕ is a bijection, then ϕ^{-1} is also a morphism and ϕ is an isometry.

Definition 1.4. Let A be a C*-algebra.

(i) A *C*-subalgebra* of A is a (topologically) closed subalgebra $B \leq A$ such that for all $x \in B$ also $x^* \in B$. In other words, B together with the restrictions of all relevant data of A becomes itself a C*-algebra.

(ii) Given a subset $S \subseteq A$, the intersection of all C*-subalgebras of A which contain S is again a C*-subalgebra. We denote it by $C^*(S)$ and say that S *generates* a C*-subalgebra B when $C^*(S) = B$. In case $S = \{x_1, \ldots, x_k\}$ is a finite set, we also write $C^*(x_1, \ldots, x_k)$ for $C^*(S)$.

In the leading example of continuous functions $X \to \mathbb{C}$, suppose that X is noncompact (but still locally compact). It is then useful to look at the continuous functions vanishing at infinity, which form the C*-subalgebra $C_0(X)$. Since we assume X to be noncompact, $C_0(X)$ does not have a unit. As for Banach algebras, one can always adjoin a unit as follows:

Definition & Proposition 1.5. Let A be a C*-algebra. Then $A^+ := A \oplus \mathbb{C}$, with the multiplication

$$(x_1, z_1)(x_2, z_2) := (x_1x_2 + z_1x_2 + z_2x_1, z_1z_2) \quad \forall x_1, x_2 \in A,\ z_1, z_2 \in \mathbb{C},$$

the norm

$$\|(x, z)\| := \sup_{y \in A, \|y\|=1} \|xy + zy\| \quad \forall x \in A, z \in \mathbb{C},$$

and the involution

$$(x, z)^* := (x^*, z^*) \quad \forall x \in A, z \in \mathbb{C}$$

is a unital C*-algebra with unit $(0, 1)$, which contains A (as $\{(x, 0) \mid x \in A\}$) as a maximal ideal. If A is commutative, then so is A^+. One calls A^+ the *unitisation* of A.

Proof. For a verification of the C*-property, see [1, p. 27]. $\square$

1.1.2 Commutative C*-algebras

As it turns out, our first example of continuous functions on a locally compact Hausdorff space already captures all commutative C*-algebras. This is known as the Gelfand-Naimark theorem[1]. It hinges on the following way of assigning a topological space to an arbitrary C*-algebra:

Definition 1.6. Let A be a C*-algebra. A *character* of A is a non-zero algebra homomorphism $A \to \mathbb{C}$. The *Gelfand spectrum* $G(A)$ is the set of all characters of A, viewed as a subspace of A^* equipped with the weak-* topology.

Assuming A has a unit, denoted $1 \in A$, we can also consider the spectra of individual elements in the usual sense. That is, given $x \in A$ we can form its spectrum

$$\sigma(x) := \{z \in \mathbb{C} \mid x - z1 \text{ is not invertible}\}.$$

Now for a character $\mu \colon A \to \mathbb{C}$ and $x \in A$ we necessarily have $\mu(x) \in \sigma(x)$ (otherwise $\ker \mu = A$ which we have excluded in the definition). Thus $|\mu(x)|$ is bounded above by the spectral radius and a fortiori by $\|x\|$. We conclude that $\|\mu\| \leq 1$, so the Gelfand spectrum $G(A)$ is contained in the unit ball of A^*. Since $G(A) \subseteq A^*$ is defined by closed conditions[2], it is also closed, hence compact by the Banach-Alaoglu theorem.

[1] There are two Gelfand-Naimark theorems in this context: the one we refer to, and another one characterising *all* C*-algebras, see e. g. [1, p.31].

[2] One can use the condition $\mu(1) = 1$ to ensure $\mu \neq 0$.

In case A is not unital, we pass to the unitisation A^+. This introduces a new character $\mu_0 \in G(A^+)$ defined by $\mu_0(x, z) := z$. All remaining characters $\mu \neq \mu_0$ must then fulfil $\mu|_A \neq 0$ and hence restrict to characters of A. In fact, this leads to a homeomorphism $G(A^+) \setminus \{\mu_0\} \cong G(A)$, so $G(A)$ is a locally compact Hausdorff space whose one-point compactification is $G(A^+)$. We summarise the preceding discussion in the following:

Proposition 1.7. *The Gelfand spectrum $G(A)$ of a C*-algebra A is a locally compact Hausdorff space. If A is unital, then $G(A)$ is compact, and otherwise its one-point compactification is $G(A^+)$.*

Starting from these observations, it is not hard to show that a locally compact Hausdorff space can be recovered from $C_0(X)$ as $X \cong G(C_0(X))$. Conversely, one can realise any C*-algebra as an algebra of continuous functions via $A \cong C_0(G(A))$, using the following construction: For $x \in A$, we form the continuous map $\hat{x} \colon G(A) \to \mathbb{C}$ given by $\hat{x}(\mu) := \mu(x)$. Varying x, we obtain a morphism of C*-algebras $A \to C_0(G(A))$, $x \mapsto \hat{x}$, called the *Gelfand transform* of A. By the following theorem, this realises A as an algebra of continuous functions (cf. [1]):

Theorem 1.8 (Gelfand–Naimark)**.** *Let A be a commutative C*-algebra. Then the* Gelfand transform

$$\mathcal{G} \colon A \to C_0(M(A)), \quad x \mapsto \hat{x},$$

is an isometric isomorphism of C-algebras.*

The above suggests that the assignments $X \mapsto C(X)$ and $A \mapsto G(A)$ are, in a certain sense, inverses. In fact, both G and C can be lifted to contravariant functors in a straightforward manner. For instance, the functor C sends morphisms (i. e. continuous maps) $f \colon Y \to X$ to

$$C(f) \colon C(X) \to C(Y), \quad h \mapsto h \circ f, \tag{1.1}$$

which is easily seen to be a morphism of C*-algebras.

These functors then indeed define an equivalence of categories between *compact* Hausdorff spaces and *unital* commutative C*-algebras.[3] One can weaken 'compact' to 'locally compact' and simultaneously drop unitality by suitably restricting the classes of morphisms on both sides (to continuous *proper* maps, and so-called *proper morphisms* (of C*-algebras), respectively).[4] [1]. To summarise, we have seen that commutative C*-algebras are equivalent to locally compact Hausdorff spaces.

1.1.3 Further general theory of C*-algebras

Here we collect various, only loosely related elements of the theory of C*-algebras to be used later.

Several basic concepts from operator theory can be formulated in the setting of C*-algebras:

[3] More precisely, contravariance dictates that one category has to be equivalent to the *opposite* category of the other.

[4] Moreover, one then takes C_0 instead of C, in accordance with Theorem 1.8.

Definition 1.9. Let A be a C*-algebra. An element $a \in A$ is called

- *normal* if $x^*x = xx^*$,
- *unitary* if A is unital and $x^*x = 1 = xx^*$,
- *self-adjoint* if $x^* = x$, and
- *positive*, denoted $x \geq 0$, if there exists an element $y \in A$ such that $x = y^*y$.

Note that a positive element is necessarily self-adjoint, and that both unitary and self-adjoint elements are also normal.

Further, $x \in A$ is normal if and only if the C*-subalgebra $C^*(x)$ is commutative. If A is unital, then $C^*(x, 1)$ will even be commutative and unital. One can then show that the Gelfand spectrum of $C^*(x, e)$ is homeomorphic to $\sigma(x)$. Via Theorem 1.8, this yields an abstract version of the *continuous functional calculus* for normal elements. Just as with linear operators, one can use this calculus to apply continuous functions on $\sigma(x)$ to x. In particular, one derives the following:

Proposition 1.10. *Let A be a C*-algebra and $x \in A$.*

(i) If x is unitary (in A^+ if A is not unital) then $\sigma(x)$ is contained in the unit circle $S^1 \subseteq \mathbb{C}$.

(ii) If x is self-adjoint then $\sigma(x) \subseteq \mathbb{R}$.

(iii) x is positive if and only if $\sigma(x) \subseteq [0, \infty)$.

We next turn to certain bounded linear functionals which respect positivity.

Definition 1.11. Let A be a C*-algebra.

(i) A *state* of A is a bounded linear functional $f \colon A \to \mathbb{C}$ of unit norm such that

$$x \geq 0 \Rightarrow f(x) \geq 0$$

for all $x \in A$.

(ii) A state $f \colon A \to \mathbb{C}$ is called *faithful* if $f(x) = 0$ implies $x = 0$ for all positive $x \in A$.

(iii) A *trace* on A is a state $t \colon A \to \mathbb{C}$ such that

$$t(xy) = t(yx) \quad \forall x, y \in A.$$

There is a notion of *ideals* for C*-algebras, which coincides with the one usually used for Banach algebras:

Definition 1.12. An *ideal* of a C*-algebra A is a closed linear subspace $I \subseteq A$ such that for all $x \in A$ and $y \in I$ we have $xy, yx \in I$.

In particular, the kernel of any morphism of C*-algebras is an ideal. Note that closure under the *-involution is not part of the definition, but follows from it (cf. [8, p.12]).

1.2 Non-commutative spaces

As laid out in subsection 1.1.2, (reasonably well-behaved) topological spaces are fully described by commutative C*-algebras. On the algebraic side of this duality, there is a clear option for generalisation, namely to drop the assumption of commutativity. At least heuristically, an arbitrary, not necessarily commutative C*-algebra should then describe a generalised kind of space. A *non-commutative space* is then simply the formal dual of a C*-algebra.[5] This point of view is described in more detail in [9].

Of course, only commutative C*-algebras can possibly consist of continuous functions into $\mathbb{C}$ as in Theorem 1.8, so defining non-commutative spaces like this initially does not seem to add anything of substance. In other words, starting out from an arbitrary non-commutative C*-algebra, the corresponding non-commutative space may well exist only formally. However, there are at least two reasons why this concept is useful:

1. In several situations one is led to interesting objects which ought to admit a topological or geometric description, but are nevertheless hard to capture classically. In [6, ch. II], Connes lists several such examples where one can instead describe the object in question by a (non-commutative) C*-algebra. A particular example is the space of leaves of a foliated manifold, which is described by a C*-algebra associated to its holonomy groupoid. In this context, one can interpret the leaf space as a non-commutative space corresponding to the C*-algebra used in its description.

2. As C*-algebras are of independent interest, the duality to spaces can also serve as a heuristic for extending topological invariants to arbitrary C*-algebras. An immediate example is K-theory (cf. e. g. [1, ch.3]), which can be applied to arbitrary C*-algebras despite being motivated by vector bundles, i. e. by classical geometric structures. The corresponding notion of non-commutative vector bundles is discussed in section 1.4. In the context of this thesis, the most important example is cyclic cohomology (see section 1.3) which can be viewed as a generalisation of de Rham homology.[6]

Non-commutative tori (to be introduced in chapter 2) in fact relate to both of the above remarks. On the one hand, we will describe and use them as C*-algebras, and use cyclic cohomology to draw conclusions about their structure. On the other hand, they are related to the leaf spaces of certain foliations of the (usual) torus.

1.3 Cyclic cohomology

In this section we closely follow the treatment in [6].

[5] The language suggests that it should be the dual of a *non-commutative* C*-algebra. Instead, one often takes 'non-commutative' to mean 'not necessarily commutative' in this context.

[6] Strictly speaking, de Rham homology and cohomology are structures on $C^\infty(X)$ (assuming X to be a smooth manifold) rather than on $C(X)$. Indeed, here one works with algebras of similar structure as $C^\infty(X)$, which may be thought of as subalgebras of "smooth functions" on a non-commutative space.

In subsection 1.1.2, we sketched the C*-algebraic description of a (locally compact Hausdorff) topological space X via its continuous functions. Let us now assume that X is a compact orientable smooth manifold. It is then natural to use the algebra $C^\infty(X)$ of smooth functions $X \to \mathbb{C}$, and one can describe various data associated to X through the lens of $C^\infty(X)$.

In particular, $C^\infty(X)$ appears as the degree zero component of the graded algebra $\Omega^*(X)$ of differential forms. Equipping $\Omega^*(X)$ with the exterior derivative d, the identity $d \circ d = 0$ leads to the well-known de Rham cohomology. There is also a dual theory of de Rham *homology*, which employs linear functionals on $\Omega^*(X)$ (called *currents*) in place of differential forms [15]. Cyclic cohomology generalises de Rham homology[7] by allowing other algebras in place of $C^\infty(X)$. [5, ch. II]

For the remainder of this section, A will denote an associative unital[8] but not necessarily commutative algebra over $\mathbb{C}$. Although the constructions of the first two subsections are purely algebraic, we will eventually need topological structure on A as well. More precisely, we assume A to be a locally convex algebra (motivated by the fact that this is true for $C^\infty(X)$).

Introducing continuity into the algebraic framework is mostly straightforward, and amounts to requiring continuity for the necessary algebraic operations (such as the multiplication in A whose continuity is part of the definition of a locally convex algebra). Aside from that, we note that all tensor products of locally convex spaces should be understood as projective tensor products.

In the guiding example of $C^\infty(X)$, where X is an n-dimensional compact orientable manifold, integration provides a linear map

$$\int : \Omega^n(X) \to \mathbb{C}, \quad \omega \mapsto \int_X \omega. \tag{1.2}$$

To connect this to the algebra $C^\infty(X)$, we can use the wedge product to extract an $(n+1)$-linear map

$$C^\infty(X)^{n+1} \to \mathbb{C}, \quad (f_0, \dots, f_n) \mapsto \int_X f_0 \, df_1 \wedge \cdots \wedge df_n. \tag{1.3}$$

As it turns out, the map (1.3) fits into the framework of Hochschild cohomology. This connection is made more precise in subsection 1.3.2, after recalling the necessary Hochschild cohomology in 1.3.1.

1.3.1 Hochschild cohomology

Definition 1.13. The *(continuous) Hochschild complex* associated to the algebra A is the cochain complex $(C^\bullet(A), b)$ of $\mathbb{C}$-vector spaces where $C^n(A)$, for $n \in \mathbb{N}$, denotes

[7] When switching between the point of view of the space X and that of the algebra $C^\infty(X)$, covariance and contravariance are interchanged, thus de Rham *homology* is related to cyclic *cohomology*.

[8] We restrict ourselves to unital algebras here because the algebras describing non-commutative tori studied in this thesis are unital. However, cyclic cohomology can also be defined for nonunital algebras. Unitality is mainly used in section 1.3.3.

the space of (continuous) $\mathbb{C}$-multilinear maps $A^{n+1} \to \mathbb{C}$, and b sends $c \in C^n(A)$ to $b(c) \in C^{n+1}(A)$ given by

$$b(c)(a_0, \dots, a_{n+1}) \coloneqq \sum_{j=0}^{n} (-1)^j c(a_0, \dots, a_j a_{j+1}, \dots a_{n+1}) + (-1)^{n+1} c(a_{n+1}a_0, a_1, \dots, a_{n+1}).$$

A straightforward calculation shows that $b \circ b = 0$. The resulting cohomology is called *Hochschild cohomology*[9] and denoted $HH^\bullet(A)$. As usual in cohomology theories, we call elements of $C^\bullet(A)$ *cochains*, and elements of $\ker b$ *cocycles*. We also use the notation $Z^n(A) \coloneqq \ker b \cap C^n(A)$ for the space of degree n cocycles.

Observe that, in the continuous setting, continuity of the multiplication in A ensures continuity of $b(c)$, and thus well-definedness of b. Similar observations will apply in what follows.

There is a more conceptual interpretation of Hochschild cohomology which we will later need. For this, one views A itself as an A-bimodule, or equivalently as a module over the *enveloping algebra* $A^e \coloneqq A \otimes_{\mathbb{C}} A^{op}$, where A^{op} denotes the opposite algebra. As an A^e-module, A has the so-called *standard (free) resolution*

$$\cdots \longrightarrow A^e \otimes A \otimes A \longrightarrow A^e \otimes A \longrightarrow A^e \xrightarrow{\ \epsilon\ } A,$$

with the A^e-module $A^e \otimes A^{\otimes n}$ in degree n, $n \in \mathbb{N}$. Here ϵ is simply multiplication $A \otimes A^{op} \to A$. Identifying $A^e \otimes A^{\otimes n}$ with $A^{\otimes(n+2)}$, the arrow at degree $n \geq 1$ is given by

$$a_0 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{j=0}^{n} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1} \in A^{\otimes(n+1)}.$$

To connect this to Hochschild cohomology, first observe that there are isomorphisms

$$C^n(A) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes(n+1)}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A^*) \cong \mathrm{Hom}_{A^e}(A^e \otimes A^{\otimes n}, A^*).$$

Explicitly, the isomorphisms above identify $c \in C^n(A)$ with the A^e-linear map $f\colon A^e \otimes A^{\otimes n} \to A^*$ given by

$$f((1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_n)(a_0) = c(a_0, a_1, \dots a_n) \quad \forall a_0, \dots, a_n \in A. \tag{1.4}$$

Applying the functor $\mathrm{Hom}_{A^e}(\cdot, A^*)$ also to the morphisms of the standard resolution, one arrives at the coboundary maps b of Hochschild cohomology. In other words, $HH^\bullet(A) = \mathrm{Ext}^\bullet_{A^e}(A, A^*)$.[10] In particular, the Hochschild cohomology of A can be computed using any projective resolution.

[9] More precisely, this would be called the Hochschild cohomology *with values in* A^*, the dual of A, but we will not need other forms of Hochschild cohomology.

[10] Here the A^e-module structure on A^* is $((a \otimes b) \cdot f)(c) = f(bca)$ for $a, b, c \in A$ and $f \in A^*$.

1.3.2 Traces and cyclicity

Let us now generalise the map $\int : \Omega^n(X) \to \mathbb{C}$ from (1.2). To this end, we first introduce some terminology:

Definition 1.14.

(i) A *(locally convex) differential graded algebra (over $\mathbb{C}$)*, or *dg algebra* for brevity, is a tuple (Ω, d) consisting of a (locally convex) graded $\mathbb{C}$-algebra $\Omega = \bigoplus_{n\in\mathbb{N}} \Omega^n$, and a (continuous) graded derivation $d\colon \Omega \to \Omega$ satisfying $d \circ d = 0$. To be a graded derivation means that d should be linear and fulfil

$$\forall k, \ell \in \mathbb{N}, \forall \omega_1 \in \Omega^k, \omega_2 \in \Omega^\ell : \quad d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^k \omega_1 d(\omega_2).$$

Given dg algebras $(\Omega_1, d_1), (\Omega_2, d_2)$ over $\mathbb{C}$, a (continuous) linear map $f\colon \Omega_1 \to \Omega_2$ is called a *morphism of differential graded algebras* if it fulfils $f \circ d_1 = d_2 \circ f$.

We will usually refer to a dg algebra (Ω, d) simply as Ω, and denote the graded derivations of all dg algebras by d whenever there is no risk of confusion.

(ii) Let $\Omega \equiv (\Omega, d)$ be a dg algebra and $n \in \mathbb{N}$. A *closed graded trace of dimension n* on Ω is a (continuous) linear map $\int : \Omega^n \to \mathbb{C}$ such that

$$\int d\omega = 0 \quad \forall \omega \in \Omega^{n-1},$$

and

$$\int \omega_1\omega_2 = (-1)^{|\omega_1||\omega_2|} \int \omega_2\omega_1 \quad \forall \omega_1 \in \Omega^i, \omega_2 \in \Omega^j \text{ with } i + j = n.$$

We will refer to the former property as *closedness* and to the latter as *graded traciality.*

(iii) A *cycle over A of dimension n* is a triple $(\Omega, \int, \rho)$ consisting of a unital dg algebra Ω with a closed graded trace $\int$ of dimension n, together with a (continuous) homomorphism $\rho\colon A \to \Omega^0$ of algebras.

The motivating example for these definitions is of course the differential graded algebra $\Omega^*(X)$ of differential forms (with the wedge product), with the integral as a closed graded trace, and ρ being just the inclusion map. Here closedness comes from Stokes' theorem, whereas graded traciality is already a consequence of graded commutativity of the wedge product.

As in (1.3), we can use a cycle over A to define a multilinear map, or in other words a Hochschild cochain:

Definition 1.15. The *character* associated to a cycle $(\Omega, \int, \rho)$ over A of dimension n is the functional $\chi \in C^n(A)$ given by

$$\chi(a_0, \dots, a_n) := \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_n) \quad \forall a_0, \dots, a_n \in A.$$

The connection to Hochschild cohomology is not just superficial:

Proposition 1.16. *Let $(\Omega, \int, \rho)$ be a cycle over A of dimension n. Then its character χ is an n-cocycle, i. e. $b(\chi) = 0$.*

Proof. Let $a_0, \ldots, a_{n+1} \in A$. Then the first n summands of $b(\chi)(a_0, \ldots, a_{n+1})$ are of the form

$$\begin{aligned}
&(-1)^j \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_j a_{j+1}) \cdots d\rho(a_{n+1}) \\
&= (-1)^j \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_j) \rho(a_{j+1}) \cdots d\rho(a_{n+1}) \\
&\quad + (-1)^j \int \rho(a_0) d\rho(a_1) \cdots \rho(a_j) da_{j+1} \cdots d\rho(a_{n+1}).
\end{aligned}$$

Due to the signs, almost all of these terms cancel and we are left with

$$\begin{aligned}
&b(\chi)(a_0, \ldots, a_{n+1}) \\
&= \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_{n+1}) + (-1)^{n+1} \int d\rho(a_0) \cdots d\rho(a_n) \rho(a_{n+1}) \\
&\quad + (-1)^{n+1} \int d\rho(a_{n+1}) \rho(a_0) d\rho(a_1) \cdots d\rho(a_n) \\
&\quad + (-1)^{n+1} \int \rho(a_{n+1}) d\rho(a_0) \cdots d\rho(a_n).
\end{aligned}$$

By graded traciality of $\int$, the first summand cancels with the third, and the second with the fourth, so we obtain $b(\chi) = 0$ as claimed. □

Moreover, characters of cycles have one further distinguishing property, called *cyclicity*:

Definition 1.17. An n-cochain $c \in C^n(A)$ is called *cyclic* if it fulfils

$$c(a_n, a_0, \ldots, a_{n-1}) = (-1)^n c(a_0, \ldots, a_n) \quad \forall a_0, \ldots, a_n \in A,$$

i. e. if it is skew-symmetric under cyclic permutations.

Proposition 1.18. *χ as in Proposition 1.16 is cyclic.*

Proof. Let $a_0, \ldots, a_n \in A$. Using closedness and graded traciality of $\int$, we compute

$$\begin{aligned}
0 &= \int d\big(\rho(a_n) \rho(a_0) d\rho(a_1) \cdots d\rho(a_{n-1})\big) \\
&= \int d\rho(a_n) \rho(a_0) d\rho(a_1) \cdots d\rho(a_{n-1}) + \int \rho(a_n) d\rho(a_0) \cdots d\rho(a_{n-1}) \\
&= (-1)^{n-1} \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_{n-1}) d\rho(a_n) + \int \rho(a_n) d\rho(a_0) \cdots d\rho(a_{n-1}) \\
&= (-1)^{n-1} \chi(a_0, \ldots, a_n) + \chi(a_n, a_0, \ldots, a_{n-1}),
\end{aligned}$$

which shows that χ is cyclic. □

Motivated by Proposition 1.18, we should aim to describe the space of all cyclic cochains. It turns out that this is a subcomplex of the Hochschild complex.

Proposition 1.19. *Let $n \in \mathbb{N}$ and $c \in C^n(A)$ be a cyclic cochain. Then $b(c) \in C^{n+1}(A)$ is also cyclic. Hence the cyclic cochains form a subcomplex of the Hochschild complex $(C^\bullet(A), b)$.*

Proof. Let $a_0, \dots, a_{n+1} \in A$. Then

$$\begin{aligned} &b(c)(a_{n+1}, a_0, \dots, a_n) \\ &= c(a_{n+1}a_0, \dots, a_n) + \sum_{j=0}^{n-1} (-1)^{j+1} c(a_{n+1}, a_0, \dots, a_j a_{j+1}, \dots, a_n) \\ &\quad + (-1)^{n+1} c(a_n a_{n+1}, a_0, \dots, a_{n-1}). \end{aligned}$$

Now recognise the first summand of the right hand side as $(-1)^{n+1}$ times the last summand of $b(c)(a_0, \dots, a_{n+1})$, and apply cyclicity of c in all other summands to get

$$\begin{aligned} &b(c)(a_{n+1}, a_0, \dots, a_n) \\ &= \sum_{j=0}^{n-1} (-1)^{j+n+1} c(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{2n+1} c(a_0, \dots, a_{n-1}, a_n a_{n+1}) + c(a_{n+1}a_0, \dots, a_n) \\ &= (-1)^{n+1} b(c)(a_0, \dots, a_{n+1}), \end{aligned}$$

so $b(c)$ is cyclic. Since cyclicity is clearly also preserved when taking linear combinations, the cyclic cochains form a subcomplex. $\square$

Definition 1.20. For $n \in \mathbb{N}$, we denote the space of (continuous) cyclic n-cochains by $C^n_\lambda(A)$, and the space of (continuous) cyclic n-cocycles by $Z^n_\lambda(A)$. The cohomology of the subcomplex $(C^\bullet_\lambda(A), b)$ is called *cyclic cohomology*, and we denote its n-th cohomology group by $HC^n(A)$.

Example 1.21. For the $\mathbb{C}$-algebra $\mathbb{C}$, we have $C^n(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^{\otimes(n+1)}, \mathbb{C}) \cong \mathbb{C}$, for every $n \in \mathbb{N}$. An explicit generator of $C^n(\mathbb{C})$ is given by the multiplication map

$$\sigma_n \colon \mathbb{C}^{n+1} \to \mathbb{C}, \quad (a_0, \dots, a_n) \mapsto a_0 \cdots a_n,$$

which is cyclic if and only if n is even. Thus

$$C^n_\lambda(\mathbb{C}) \cong \begin{cases} \mathbb{C} & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases}$$

which immediately yields

$$HC^n(\mathbb{C}) \cong \begin{cases} \mathbb{C} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \tag{1.5}$$

Although we will not need much of the framework of category theory, it is useful to observe that these cohomology theories are functorial:

Proposition 1.22. *Both Hochschild and cyclic cohomology are functorial. More precisely, for all $n \in \mathbb{N}$ the functor HH^n sends a (continuous) homomorphism $f\colon A \to B$ between (locally convex) algebras A, B to the group homomorphism*

$$f^*\colon HH^\bullet(B) \to HH^\bullet(A), \quad HH^n(B) \ni c \mapsto c \circ f^{\times(n+1)},$$

and f^* commutes with the inclusions $HC^n \hookrightarrow HH^n$. In other words, the inclusion $HC^n \hookrightarrow HH^n$ is a natural transformation.

Proof. Let A, B, C be (locally convex) algebras, $f\colon A \to B$, $g\colon B \to C$ (continuous) homomorphisms. Given $c \in C^n(B)$, we have

$$f^*(b(c)) = b(c) \circ f^{\times(n+2)} = b(c \circ f^{\times(n+1)}) = b(f^*c)$$

by definition of b, so f^* really defines a homomorphism $HH^n(B) \to HH^n(A)$.

Of course,

$$(g \circ f)^*(c) = c \circ (g \circ g)^{\times(n+1)} = c \circ g^{\times(n+1)} \circ f^{\times(n+1)} = (f^* \circ g^*)(c),$$

and $(\mathrm{id}_B)^*(c) = c \circ (\mathrm{id}_B)^{\times(n+1)} = c$, proving functoriality.

Lastly, if c is in fact cyclic, then for $b_0, \dots, b_n \in B$ we compute

$$\begin{aligned}(f^*c)(b_n, b_0, \dots, b_{n-1}) &= c(f(b_n), f(b_0), \dots, f(b_{n-1}))\\ &= (-1)^n c(f(b_0), \dots, f(b_n))\\ &= (-1)^n (f^*c)(b_0, \dots, b_n),\end{aligned}$$

so f^*c is cyclic as well. □

So far we have seen that every cycle over A defines a cyclic cocycle, namely its character. This map

$$\{\text{cycles } (\Omega, \textstyle\int, \rho)\} \to \{\text{cyclic cocycles}\}$$

is in fact surjective, i. e. every cyclic cocycle over A is the character of some cycle. The proof of this statement uses the *universal dg algebra* ΩA over A, which we now introduce.

Definition 1.23. Let A be a (locally convex) $\mathbb{C}$-algebra. The *universal differential graded algebra* over A is the (locally convex) dg algebra $(\Omega A, d)$, defined as follows:

1. As vector spaces, we set $\Omega^0 A := A$, and

   $$\Omega^n A := A^+ \otimes A^{\otimes n}, \quad n \geq 1,$$

   where A^+ denotes the algebra $A \oplus \mathbb{C}$ of A, with the same product and inclusion $A \hookrightarrow A^+$ as in Definition 1.5. This unitisation is formed even if A itself is already unital.[11] We identify the ideal $A \oplus 0 \subset A^+$ with A.

[11] Rather than as unitisation, one can interpret the added copy of $\mathbb{C}$ as “providing space” for the derivation d, defined below. Since A is itself unital (which is not needed in the context of ΩA), an analogous construction is possible using A and $A/\mathbb{C}e$ in place of A^+ and A, respectively (cf. [1][3]). We use the one presented here because it avoids complications in Proposition 1.25 below.

2. We denote an element $\alpha \otimes a_1 \otimes \cdots \otimes a_n \in \Omega^n A$ as $\alpha da_1 \dots da_n$, where $\alpha \in A^+$ for $n \geq 1$ and $\alpha \in A \subset A^+$ for $n = 0$. So far this is just notation. Now writing $\alpha = (a_0, z)$, we define the map $d\colon \Omega A \to \Omega A$ by

$$d\big((a_0, z)da_1 \dots da_n\big) \coloneqq eda_0 da_1 \dots da_n \in \Omega^{n+1} A,$$

where $e = (0, 1) \in A^+$ denotes the unit.

3. Finally, for $m, n \in \mathbb{N}$, we define the product of $\alpha da_1 \dots da_m \in \Omega^m A$ and $\beta db_1 \dots db_n \in \Omega^n A$, with $\beta = (b_0, \mu)$, as

$$\begin{aligned}
&(\alpha da_1 \dots da_m)(\beta db_1 \dots db_n) \coloneqq \\
&\quad \alpha da_1 \dots da_{m-1} d(a_m \beta) db_1 \dots db_n \\
&\quad + \sum_{j=1}^{m-1} (-1)^{m-j} \alpha da_1 \dots da_{j-1} d(a_j a_{j+1}) \dots da_m db_0 \dots db_n \\
&\quad + (-1)^m \alpha\beta da_1 \dots db_n.
\end{aligned} \tag{1.6}$$

Here the terms $d(a_m \beta)$, $d(a_j a_{j+1})$ should be understood in terms of the notation of item 2 rather than an application of d so that the definition is not circular.[12]

Remark 1.24.

(i) With the full definition of ΩA in place, we can reinterpret the suggestive notation $\alpha da_1 \dots da_n$. Firstly, we have $a_1 \in A \subset A^+$ with $d(a_1) = eda_1$, so

$$\alpha d(a_1) = \alpha \cdot (eda_1) = \alpha da_1.$$

Continuing from this, we get

$$(\alpha d(a_1))d(a_2) = (\alpha da_1)(eda_2) = \alpha da_1 da_2,$$

where we used that $e = (0, 1)$ (so almost all terms in the product formula vanish), and repeating this step eventually yields

$$\alpha \cdot d(a_1) \cdots d(a_n) = \alpha da_1 \dots da_n.$$

This reconciles the two interpretations of the term $\alpha da_1 \dots da_n$, and explains the notation.

(ii) The somewhat complicated formula (1.6) defining the product can be seen as a consequence of (or necessary condition for) associativity and the graded derivation property. That is, if we want d to be a graded derivation, we must have

$$da_m \beta = d(a_m \beta) - a_m d\beta.$$

[12]Note that $a_m \beta = (a_m b_0 + \mu a_m, 0) \in A \subset A^+$ so we can interpret $a_m \beta$ as an element of A.

Assuming this identity and part (i) of this remark, we can move the factor β past each of the da_j, which leads to the formula. As a consequence, the graded derivation property of d is built into the product. It is also clear that $d \circ d = 0$, so only associativity of the product has to be checked (by a simple computation) to conclude that ΩA really is a dg algebra.

(iii) ΩA is universal in the following sense: given any dg algebra Ω with an algebra homomorphism $\phi\colon A \to \Omega^0$, there exists a unique morphism of dg algebras $\tilde{\phi}\colon \Omega A \to \Omega$ such that $\tilde{\phi} \circ \iota = \phi$, where $\iota\colon A \to \Omega A$ is the inclusion as $\Omega^0 A$.[13] [1]

The universal dg algebra provides a way of assigning a closed graded trace to any cyclic cocycle:

Proposition 1.25. *For $n \in \mathbb{N}$, let $c \in Z^n_\lambda(A)$ be a cyclic cocycle. Then there exists a unique closed graded trace $\hat{c}$ of dimension n on ΩA whose character is c.*

Proof. Requiring that c be the character of $\hat{c}$ translates to

$$\hat{c}(a_0 da_1 \dots da_n) = c(a_0, \dots, a_n), \quad a_0, \dots, a_n \in A,$$

which uniquely defines $\hat{c}$ on the subspace $A^{\otimes(n+1)}$ of $\Omega^n A = A^+ \otimes A^{\otimes n}$. Its complement $\mathbb{C} \otimes A^{\otimes n}$ consists of elements of the form

$$z da_1 \dots da_n = d(z a_1 da_2 \dots da_n), \quad z \in \mathbb{C},\ a_1, \dots, a_n \in A, \tag{1.7}$$

which must be sent to zero if $\hat{c}$ is to be closed. Hence $\hat{c}$ is uniquely determined as

$$\hat{c}((a_0, z) da_1 \dots da_n) = c(a_0, \dots, a_n), \quad z \in \mathbb{C},\ a_0, \dots, a_n \in A. \tag{1.8}$$

Now (1.8) clearly defines a linear functional on $\Omega^n A$, which is closed because all elements of $d(\Omega^{n-1} A)$ are of the form described in (1.7). Graded traciality is shown by a similar computation as in Proposition 1.18. □

Example 1.26. The universal dg algebra $\Omega\mathbb{C}$ over the $\mathbb{C}$-algebra $\mathbb{C}$ has $\Omega^0\mathbb{C} = \mathbb{C}$ and $\Omega^n\mathbb{C} = \mathbb{C}^+ \otimes \mathbb{C}^{\otimes n} \cong (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^{\otimes n}$ for $n \geq 1$. If $n \geq 2$ is even, the generator σ_n of C^n_λ found in Example 1.21 is the character of the closed graded trace

$$\widehat{\sigma_n}\colon \Omega^n\mathbb{C} \to \mathbb{C}, \quad (w_0, z) \otimes w_1 \otimes \dots \otimes w_n \mapsto w_0 w_1 \cdots w_n, \quad z, w_0, \dots, w_n \in \mathbb{C}.$$

It will be useful to extend the correspondence established above to general cochains:

Definition 1.27. Let $n \in \mathbb{N}$.

(i) For $c \in C^n(A)$, define $\hat{c}\colon \Omega^n A \to \mathbb{C}$ by

$$\hat{c}((a_0, z) da_1 \dots da_n) \coloneqq c(a_0, \dots, a_n) \quad z \in \mathbb{C},\ a_0, \dots, a_n \in A.$$

[13] This universal property of ΩA explains why it is suitable for the construction in Proposition 1.25.

(ii) Given a dg algebra Ω, an algebra homomorphism $\rho\colon A \to \Omega^0$, and a linear functional $f\colon \Omega^n \to \mathbb{C}$, define the *character* of f as the Hochschild cochain

$$(a_0, \ldots, a_n) \mapsto f(\rho(a_0)d\rho(a_1)\cdots d\rho(a_n)), \quad z \in \mathbb{C},\, a_0, \ldots, a_n \in A.$$

This extends Proposition 1.25 in the sense that any Hochschild cochain c is represented as the character of $\hat{c}$ (which however will generally not be a closed graded trace).

1.3.3 Relating Hochschild and cyclic cohomology

In this subsection we discuss an exact sequence

$$\cdots \longrightarrow HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} \cdots$$

relating Hochschild and cyclic cohomology. Since Hochschild cohomology tends to be more accessible than cyclic cohomology (e. g. via an explicit projective resolution of A), this is a valuable tool for computing cyclic cohomology. We start by describing the operators I, B, S.

The simplest of the three is I, which is induced by the subcomplex inclusion $C_\lambda^\bullet(A) \hookrightarrow C^\bullet(A)$ (as established in Proposition 1.19). That is, I is the degree zero linear map $HH^\bullet(A) \to HC^\bullet(A)$ which sends $[c] \in HC^n(A)$ to $[c] \in HH^n(A)$.

We now turn to the operator B. For a Hochschild n-cochain $c \in C^n(A)$, we define $\lambda(c) \in C^n(A)$ as

$$\lambda(c)(a_0, \ldots, a_n) \coloneqq (-1)^n c(a_1, \ldots, a_n, a_0), \quad a_0, \ldots, a_n \in A$$

(and $\lambda(c) = c$ if $n = 0$), thus obtaining a graded module morphism $\lambda\colon C^\bullet(A) \to C^\bullet(A)$. Clearly, $C_\lambda^n(A) = \ker(\mathrm{id} - \lambda) \cap C^n(A)$. Moreover, we observe that $\lambda^{n+1}|_{C^n(A)} = \mathrm{id}_{C^n(A)}$.

Using λ, we may express the *cyclic antisymmetrisation* of $c \in C^n(A)$ as

$$P_\lambda(c) \coloneqq \sum_{j=0}^{n} \lambda^j(c) \in C_\lambda^n(A).$$

Indeed, we have

$$(\mathrm{id} - \lambda)P_\lambda(c) = (\mathrm{id} - \lambda^{n+1})(c) = 0,$$

which verifies that $P_\lambda(c) \in C_\lambda^n(A)$. On $C_\lambda^n(A)$, P_λ acts as $(n+1)\,\mathrm{id}$, so it is a multiple of a projection.

The last piece needed for forming B is the insertion operator $u\colon C^n(A) \to C^{n-1}(A)$ defined by

$$u(c)(a_0, \ldots, a_n) \coloneqq c(1, a_0, \ldots, a_n), \quad c \in C^{n+1}(A),\, a_0, \ldots, a_n \in A$$

if $n \geq 1$ and as zero on $C^0(A)$. It is here that we need the assumption that A be unital.

We now define B on the level of cochains as

$$B\colon C^n(A) \to C_\lambda^{n-1}(A), \quad B \coloneqq P_\lambda \circ u \circ (\mathrm{id} - \lambda). \tag{1.9}$$

Proposition 1.28. *The degree -1 operator* $B\colon C^\bullet(A) \to C^\bullet_\lambda(A)$ *defined above fulfils* $B \circ b = -b \circ B$. *In particular, it descends to a well-defined operator* $B\colon HH^\bullet(A) \to HC^\bullet(A)$.

Proof. Let $n \in \mathbb{N}$, $c \in C^n(A)$, and $a_0, \dots, a_{n+1} \in A$.

It will be useful to express the Hochschild coboundary operator b in terms of λ. To this end, we define $\beta(c) \in C^{n+1}(A)$ as

$$\beta(c)(a_0, \dots, a_{n+1}) \coloneqq c(a_0 a_1, a_2, \dots, a_{n+1}),$$

and extend β analogously to arbitrary Hochschild cochains. Then we can write $b(c)$ as

$$b(c) = \sum_{j=0}^{n+1} \lambda^j \beta \lambda^{-j}(c), \tag{1.10}$$

which again generalises to arbitrary cochains. Moreover, observe that $u \circ \beta = \mathrm{id}$.

If $n = 0$, then $u(c) = 0$ and hence $bB(c) = 0$. In this case, we have to show that also $Bb(c) = 0$, and this follows from

$$ub(c)(a_0) = c(1a_0) - c(a_0 1) = 0.$$

For the remainder we assume $n \geq 1$, and prove the claim in steps.

Firstly, (1.10) readily leads to the formula

$$\lambda b(c) = b\lambda(c) + \beta(\mathrm{id} - \lambda)(c), \tag{1.11}$$

and thus to

$$(\mathrm{id} - \lambda) b(c) = b(\mathrm{id} - \lambda)(c) - \beta(\mathrm{id} - \lambda)(c). \tag{1.12}$$

Secondly, let $c' \in C^n(A)$. Then

$$\begin{aligned} bu(c')(a_0, \dots, a_n) = {} & c'(1, a_0 a_1, a_2, \dots, a_n) - c'(1, a_0 a_1, a_2, \dots, a_n) + \cdots \\ & \cdots + (-1)^{n-1} c'(1, a_0, \dots, a_{n-1} a_n) + (-1)^n c'(1, a_n a_0, \dots, a_{n-1}), \end{aligned}$$

whereas

$$\begin{aligned} ub(c')(a_0, \dots, a_n) = {} & c'(a_0, a_1, \dots, a_n) - c'(1, a_0, a_2 a_3, \dots, a_n) + \cdots \\ & \cdots + (-1)^n c'(1, a_0, \dots, a_{n-1} a_n) + (-1)^{n+1} c'(a_n, a_0, \dots, a_{n-1}), \end{aligned}$$

and we can express this as

$$ub(c') = -bu(c') + (\mathrm{id} - \lambda^{-1} + \lambda^{-1} \beta u)(c'). \tag{1.13}$$

Finally, given $c'' \in C^{n-1}(A)$, (1.11) inductively leads to

$$\lambda^j b(c'') = b\lambda^j(c'') + \sum_{k=0}^{j-1} \lambda^k \beta(\mathrm{id} - \lambda) \lambda^{j-k-1}(c''), \quad j \in \mathbb{N},$$

and thus to

$$
\begin{aligned}
P_\lambda b(c'') &= \sum_{j=0}^{n} \lambda^j b(c'') \\
&= \sum_{j=0}^{n} \left(b\lambda^j(c'') + \sum_{k=0}^{j-1} \lambda^k \beta(\mathrm{id} - \lambda)\lambda^{j-k-1}(c'') \right) \\
&= bP_\lambda(c'') + b(c'') + \sum_{j=0}^{n}\sum_{k=0}^{j-1} \lambda^k \beta(\mathrm{id} - \lambda)\lambda^{j-k-1}(c''),
\end{aligned}
$$

where the last equality stems from the fact that P_λ has one less summand on $C^{n-1}(A)$ than on $C^n(A)$, and the redundant summand is precisely $b\lambda^n(c'') = b(c'')$. Now the rightmost term can be expanded as

$$
\begin{aligned}
\sum_{j=0}^{n}\sum_{k=0}^{j-1} \lambda^k \beta(\mathrm{id} - \lambda)\lambda^{j-k-1}(c'') &= \sum_{j=0}^{n}\sum_{k=0}^{j-1} (\lambda^k\beta\lambda^{j-k-1} - \lambda^k\beta\lambda^{j-k})(c'') \\
&= \sum_{k=0}^{n-1}\sum_{j=k+1}^{n} (\lambda^k\beta\lambda^{j-k-1} - \lambda^k\beta\lambda^{j-k})(c'') \\
&= \sum_{k=0}^{n-1} (\lambda^k\beta - \lambda^k\beta\lambda^{n-k})(c'') \\
&= \big(\sum_{k=0}^{n} \lambda^k\beta - \sum_{k=0}^{n} \lambda^k\beta\lambda^{-k}\big)(c'') \\
&= P_\lambda\beta(c'') - b(c''),
\end{aligned}
$$

and we obtain

$$
P_\lambda b(c'') = bP_\lambda(c'') + P_\lambda\beta(c''). \tag{1.14}
$$

Now by (1.12), we have

$$
Bb(c) = P_\lambda ub(\mathrm{id} - \lambda)(c) - \underbrace{P_\lambda u\beta(\mathrm{id} - \lambda)}_{=0}(c) = P_\lambda ub(\mathrm{id} - \lambda)(c),
$$

where we have used that $u\beta = \mathrm{id}$ and $P_\lambda(\mathrm{id} - \lambda) = 0$. Applying (1.13) with $c' := (\mathrm{id} - \lambda)(c)$, we get

$$
\begin{aligned}
P_\lambda ub(\mathrm{id} - \lambda)(c) &= -P_\lambda bu(\mathrm{id} - \lambda)(c) + P_\lambda(\mathrm{id} - \lambda^{-1} + \lambda^{-1}\beta u)(c') \\
&= -P_\lambda bu(\mathrm{id} - \lambda)(c) + P_\lambda\beta u(c'),
\end{aligned}
$$

where the second equality comes from $P_\lambda(\mathrm{id} - \lambda^{-1}) = 0$ and $P_\lambda\lambda^{-1} = P_\lambda$. Lastly, (1.14) with $c'' = u(\mathrm{id} - \lambda)(c)$ yields

$$
-P_\lambda bu(\mathrm{id} - \lambda)(c) = -bB(c) - P_\lambda\beta u(c'),
$$

and altogether we have

$$Bb(c) = -bB(c) + P_\lambda \beta u(c') - P_\lambda \beta u(c') = -bB(c)$$

as claimed. □

The third operator in the sequence is S, which is defined using an operation called the *cup product*. Given an m-cochain $c_1 \in C^m(A)$, and another (locally convex) algebra B with an n-cochain $c_2 \in C^n(B)$, we may represent them by the functionals $\widehat{c_1}$ on $\Omega^m A$ and $\widehat{c_2}$ on $\Omega^n B$, as defined in Definition 1.27. As with any pair of dg algebras, we can take the graded tensor product

$$\Omega A \otimes \Omega B = \bigoplus_{k\in\mathbb{N}} \bigoplus_{j=0}^{k} \Omega^j A \otimes \Omega^{k-j} B,$$

which becomes a differential graded algebra via the multiplication

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') := (-1)^{|\beta||\alpha'|}(\alpha\alpha') \otimes (\beta\beta') \tag{1.15}$$

and derivation

$$d(\alpha \otimes \beta) := (d\alpha) \otimes \beta + (-1)^{|\alpha|}\alpha \otimes (d\beta),$$

where $\alpha, \alpha' \in \Omega A$, $\beta, \beta' \in \Omega B$ are homogeneous elements and $|\cdot|$ denotes the degree. Now the tensor product $\widehat{c_1} \otimes \widehat{c_2}$ (followed by multiplication $\mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$) can be extended by zero to the whole degree $m+n$ component of $\Omega A \otimes \Omega B$. Its character is, by definition, the *cup product* $c_1 \# c_2 \in C^{m+n}(A \otimes B)$. We collect some facts about this operation:

Proposition 1.29.

(i) The cup product is $\mathbb{C}$-bilinear and continuous.

(ii) The cup product of two cyclic cocycles is again a cyclic cocycle.

(iii) The cup product defines a homomorphism

$$HC^m(A) \otimes HC^n(B) \to HC^{m+n}(A \otimes B), \quad [c_1] \otimes [c_2] \mapsto [c_1 \# c_2].$$

Proof. Bilinearity readily follows from bilinearity of the tensor product operation, and since all steps involved in its construction are continuous, so is the cup product. To show (ii), one verifies that the tensor product of two closed graded traces is again a closed graded trace, and then applies Propositions 1.16 and 1.18. For (iii), see Theorem III.1.12 in [6]. □

Now by Example 1.21, we have $HC^2(\mathbb{C}) \cong \mathbb{C}$ with generator $\sigma := \sigma_2 \in C^2_\lambda(\mathbb{C})$. Using the cup product, we obtain a map

$$\tilde{S} \colon C^n_\lambda(A) \to C^{n+2}(A \otimes \mathbb{C}), \quad c \mapsto \sigma \# c.$$

We can interpret $\sigma\#c$ as an element of $C^{n+2}(A)$ using the isomorphism

$$\iota\colon A \xrightarrow{\cong} \mathbb{C}\otimes A, \quad a \mapsto 1\otimes a,$$

and define

$$S\colon C^n(A) \to C^{n+2}_\lambda(A), \quad S := \tfrac{1}{n+3}P_\lambda \circ \iota^* \circ \tilde{S}.$$

Note that the projection $\frac{1}{n+3}P_\lambda$ becomes redundant when we restrict S to cyclic cocycles; this is part of Proposition 1.29(iii). Moreover, it implies that S descends to an operator $S\colon HC^n(A) \to HC^{n+2}(A)$. [14]

Via some computations in the dg algebra $\Omega\mathbb{C}$ one obtains the following explicit formula of $S(c)$ for $c \in Z^n_\lambda(A)$ in terms of $\hat{c}$ (cf. the proof of Corollary III.1.13 in [6]):

$$S(c)(a_0,\dots,a_{n+2}) = \sum_{j=1}^{n+1} \hat{c}(a_0 da_1 \dots da_{j-1} a_j a_{j+1} da_{j+2} \dots da_{n+2}) \tag{1.16}$$

for all $a_0,\dots,a_{n+2} \in A$.

With the definitions of I, B, S in place, we come to the exact sequence connecting them.

Theorem 1.30. *The operators I, B, S (on the level of cohomology) form an exact triangle*

$$\begin{tikzcd} HC^\bullet(A) \arrow[rr, "S"] & & HC^\bullet(A) \arrow[dl, "I"] \\ & HH^\bullet(A) \arrow[ul, "B"] & \end{tikzcd} \tag{1.17}$$

In other words, there is an exact sequence starting with

$$0 \longrightarrow HC^0(A) \xrightarrow{I} HH^0(A) \longrightarrow 0 \longrightarrow HC^1(A) \xrightarrow{I} \cdots$$

and of the general form

$$\cdots \longrightarrow HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} \cdots$$

for $n \geq 1$.

Proof. We outline the steps of the proof given in [6]. In this proof, all references to [6] refer to section 1 of its third chapter, where the present theorem is listed as Theorem 26. We sometimes use upper indices for the maps b, B to keep track of cochain dimension.

Firstly, it is clear from the definition of B that $B \circ I = 0$ already on the cochain level, so $\operatorname{im} I \subseteq \ker B$; the converse inclusion will be shown later.

[14] Often S is defined only on cyclic cocycles. However, it is occasionally useful to extend the definition to arbitrary Hochschild cochains as we have done it here, at the cost of having to include the projection $\frac{1}{n+3}P_\lambda$.

It is also relatively straightforward to show that $I \circ S = 0$ in cohomology. This amounts to proving that the image under S of a cyclic cocycle is a Hochschild coboundary, as shown in [6], Proposition 15. The proof relies on formula (1.16) for S, obtained from the definition of the cup product and some calculations in the dg algebra $\Omega\mathbb{C}$.

To go further requires some intermediate results. Let $n \in \mathbb{N}$, then:

1. B is surjective on the cochain level, i. e. $B(C^n(A)) = C_\lambda^{n-1}$. This is verified in [6], Corollary 20 by explicitly constructing an element in the preimage of each cyclic cochain.

2. If $c \in C^n(A)$ fulfils $b(c) \in C_\lambda^{n+1}(A)$, then $B(c) \in Z_\lambda^{n-1}(A)$ and $SB(c) = n(n+1)b(c)$ in $HC^{n+1}(A)$; this is Lemma 23 of [6]. The first assertion follows from the identity $B \circ b = -b \circ B$ of Proposition 1.28 (in this document). The proof of the second assertion relies on the fact, introduced above, that S maps cyclic cocycles to Hochschild coboundaries, as well as a substantial amount of computation.

3. The definition of B readily implies that $B \circ B = 0$ (on the cochain level), so $\operatorname{im} B^n \subseteq \ker B^{n-1}$. Moreover, the identity $b \circ B = -B \circ b$ implies that

$$b^{n-1}(\operatorname{im} B^n) \subseteq \operatorname{im} B^{n+1} \cap \ker b^n, \quad b^{n-1}(\ker B^{n-1}) \subseteq \ker B^n \cap \ker b^n.$$

It follows that there is a map (acting trivially on representatives)

$$(\operatorname{im} B^{n+1} \cap \ker b^n)/b^{n-1}(\operatorname{im} B^n) \to (\ker B^n \cap \ker b^n)/b^{n-1}(\ker B^{n-1}), \qquad (1.18)$$

and it is shown in [6], Lemma 25 that this is in fact an isomorphism. The proof uses ideas from the proof of the previous item.

Combining items 1 and 3, we obtain $\ker B \subseteq \operatorname{im} I$ as follows: suppose $c \in Z^n(A)$ with $B(c) \in b(Z_\lambda^{n-2}(A))$. By surjectivity of B on the cochain level, there exists $c' \in C^{n-1}(A)$ such that $B(c) = bB(c') = -Bb(c')$. Then $c+b(c')$ is contained in $\ker B^n \cap \ker b^n$. Now by surjectivity of the map in (1.18), there exists $c'' \in \operatorname{im} B^{n+1} \cap \ker b^n$ such that $c+b(c')-c'' \in \operatorname{im} b^{n-1}$, that is $c = c''$ in cohomology. But by item 1 we have $\operatorname{im} B^{n+1} \cap \ker b^n = Z_\lambda^n(A)$, so c'' is cyclic and thus $c \in \operatorname{im} I$. We conclude that $\operatorname{im} I = \ker B$.

Next, let $c \in B(Z^n(A)) = Z_\lambda^{n-1}(A)$, say $c = B(c')$ for $c' \in C^n(A)$. We would like to use item 2 above, for which $b(c')$ would have to be cyclic. In general we only have $b(c') \in \ker b^n \cap \ker B^n$ (since $Bb(c') = -bB(c') = -b(c) = 0$), but by item 3 there exists $c'' \in Z_\lambda^{n+1}(A) = \operatorname{im} B^{n+2}$ such that $b(c') - c'' \in b^n(\ker B^n)$, say $b(c') = c'' + b(f)$, $f \in \ker B^n$. Then $B(c' - f) = B(c') = c$, and since $b(c' - f)$ is cyclic we may apply item 2. This yields

$$S(c) = SB(c' - f) = n(n+1)b(c' - f) = n(n+1)c''. \qquad (1.19)$$

We thus have $S(c) = 0$ in $HC^{n+1}(A)$ if and only if c'' is contained in $b(Z_\lambda^n(A)) = b^n(\operatorname{im} B^{n+1})$. Since the map from item 2 is injective, this is equivalent to $b(c') \in b^n(\ker B^n)$, i. e. to there being $h \in \ker B^n$ such that $c' - h \in Z^n(A)$. But due to

$B(c' - h) = B(c') = c$, this is in turn equivalent to $c \in B(Z^n(A))$, so we have shown that $\ker S = \operatorname{im} B$.

Lastly, let $c \in Z_\lambda^{n+1}(A)$ such that $I(c)$ vanishes in cohomology. This amounts to the existence of $c' \in Z^n(A)$ such that $c = b(c')$. But then we may apply item 2 to c' and find that

$$SB(c') = n(n+1)b(c') = n(n+1)c.$$

If $n \geq 1$, we may rewrite this as $c = \frac{1}{n(n+1)}S(B(c'))$, and conclude $\operatorname{im} S = \ker I$. In the remaining case of $n = 0$, S is the zero map and exactness follows from the fact that $I\colon HC^0(A) \to HH^0(A)$ is an isomorphism (since cyclicity is vacuous in degree zero). □

Remark 1.31.

(i) The identity (1.19) in the above proof can be expressed as

$$S = n(n+1)bB^{-1}\colon HC^{n-1}(A) \to HC^{n+1}(A). \tag{1.20}$$

Indeed, we started with $c \in Z_\lambda^{n-1}(A)$ and obtained c', f such that $B(c'-f) = c$, and $S(c) = n(n+1)b(c' - f)$. Now,even though c may have several preimages under B, replacing c' by $c' + g$ with $g \in \ker B^n$, $b(c' + g)$ lands in the same equivalence class as $b(c')$ modulo $b^n(\ker(B^n)$. In other words, the class of $b(c')$ modulo $b^n(\ker(B^n))$ depends only on c, and hence so does the class of c'' modulo $b^n(\operatorname{im} B^{n+1})$. But $\operatorname{im} B^{n+1} = C_\lambda^n(A)$, so c'' is unique up to cyclic cohomology.

(ii) The operator S is called the *periodicity operator*. To explain the name, suppose there exists $N \in \mathbb{N}$ such that $HH^n(A) = 0$ for all $n \geq N$. Then, according to Theorem 1.30, S restricts to isomorphisms $HC^{n-1}(A) \xrightarrow{\cong} HC^{n+1}(A)$ for all $n \geq N$. In other words, cyclic cohomology becomes eventually periodic.

With this periodicity in mind, one makes the following definition:

Definition 1.32. The *periodic cyclic cohomology* of A is

$$HP(A) \coloneqq HC^{ev}(A) \oplus HC^{odd}(A),$$

where the *even* cyclic cohomology $HC^{ev}(A)$ and the *odd* cyclic cohomology $HC^{odd}(A)$ are defined as the direct limits of

$$HC^0(A) \xrightarrow{S} HC^2(A) \xrightarrow{S} \cdots \xrightarrow{S} HC^{2n}(A) \xrightarrow{S} \cdots$$

and

$$HC^1(A) \xrightarrow{S} HC^3(A) \xrightarrow{S} \cdots \xrightarrow{S} HC^{2n+1}(A) \xrightarrow{S} \cdots,$$

respectively.

In other words, periodic cyclic cohomology is obtained from $HC^\bullet(A)$ by identifying $S(c)$ with c for every $c \in HC^\bullet(A)$.

1.3.4 A spectral sequence for periodic cyclic cohomology

Theorem 1.30 states that the operators S, I, B (on the cohomology level) form an *exact couple*. In particular, they give rise to a spectral sequence via a standard procedure which we summarise here for convenience, based on the presentation in [11].

Definition 1.33.

(i) An *exact couple* (D, E, i, j, k) consists of two modules[15] D, E together with morphisms $i\colon D \to D$, $j\colon D \to E$, $k\colon E \to D$ such that $\operatorname{im} i = \ker j$, $\operatorname{im} j = \ker k$ and $\operatorname{im} k = \ker i$; i.e. the following triangle is exact:

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

(ii) Let (D, E, i, j, k) be an exact couple. The *derived couple* (D', E', i', j', k') consists of

- $D' := i(D)$,
- $E' := \ker d / \operatorname{im} d$, where $d := j \circ k$,
- $i' := i|_{i(D)}\colon D' \to D'$,
- $j'\colon D' \to E'$ induced by j via $j'(i(x)) := j(x) + dE$ for $x \in D$,
- $k'\colon E' \to D'$ induced by k via $k'(e + dE) := k(e)$ for $e \in E$.

Definition & Proposition 1.34. Let $(D_1, E_1, i_1, j_1, k_1)$ be an exact couple. Then the derived couple $(D'_1, E'_1, i'_1, j'_1, k'_1)$ is again exact. For $r \in \mathbb{N}$ we recursively define $(D_r, E_r, i_r, j_r, k_r)$ as the derived couple of $(D_{r-1}, E_{r-1}, i_{r-1}, j_{r-1}, k_{r-1})$ for $r \geq 2$, and set $d_r := j_r \circ k_r$. We call the sequence $(E_r, d_r)_{r\in\mathbb{N}}$ the *spectral sequence associated to the exact couple* $(D_1, E_1, i_1, j_1, k_1)$.

Proof. For the (straightforward) proof of exactness of the derived couple see Proposition 2.7 in [11]. □

Remark 1.35.

1. In light of the preceding definition one can interpret an exact couple as a module E with boundary operator d (i.e. $d \circ d = 0$), together with auxiliary data used to form the derived couple.

2. $(E_r, d_r)_r$ is a spectral sequence in the sense that $E_{r+1} = H(E_r, d_r)$ for all r. We do not yet (need to) assume E to be bigraded.

[15] over some fixed ground ring (A in our case)

Usually, a spectral sequence is expected to come with a bigrading. To implement this, assume that the modules D_1 and E_1 in the above proposition are bigraded (by $\mathbb{Z} \oplus \mathbb{Z}$) and that

- i has bidegree $(1, -1)$,
- j has bidegree $(0, 0)$,
- k has bidegree $(0, 1)$.

Then, for each r, both D_r and E_r inherit bigradings from E_{r-1}, D_{r-1}, and one inductively shows that d_r has bidegree $(1 - r, r)$ (see e. g. [11], but note the swapped indices).[16]

We may now apply this to the exact couple $(HC^\bullet(A), HH^\bullet(A), S, I, B)$ equipped with the bigradings[17]

$$(HC(A))^{p,q} := HC^{p-q}(A), \quad (HH(A))^{p,q} := HH^{p-q}(A) \quad \forall p, q \in \mathbb{Z}.$$

Then S, I, B indeed fulfil the bidegree requirements and we obtain a cohomological spectral sequence, starting with $(HH^\bullet(A), I \circ B)$. However, it is not immediately clear if, and to which module, the spectral sequence should converge. Moreover, the bigrading seems rather artificial. For an explanation it will be helpful to introduce another point of view. The remainder of this section closely follows [6] (in particular Theorem 29 of section III.1).

In Proposition 1.28, we saw that the degree -1 operator $B\colon C^\bullet(A) \to C^\bullet_\lambda(A) \subseteq C^\bullet(A)$ fulfils $B \circ B = 0$ and $B \circ b = -b \circ B$. Thus B and the Hochschild coboundary operator

[16] Our conventions here differ from the perhaps more often used convention of bidegree $(r, 1 - r)$ for cohomological spectral sequences. (This is reflected already in the bidegree assumptions on i, j, k.) One reason for this is that the spectral sequence we are interested in can be identified (see below) with the second filtration spectral sequence of a certain cohomological double complex. For such spectral sequences, $(1 - r, r)$ is the natural bidegree (in the context of usual conventions).

[17] Strictly speaking, these are not bigradings *of* $HH^\bullet(A)$ and $HC^\bullet(A)$ (if a bigrading is understood as a direct sum decomposition) but rather of infinite direct sums of them. However, this makes little difference here and we will still refer to them that way.

b fit into a double complex as shown below:

$$\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{n-2}(A) \arrow[r, "b"] \arrow[u] & C^{n-1}(A) \arrow[r, "b"] \arrow[u] & C^{n}(A) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{n-1}(A) \arrow[r, "b"] \arrow[u, "B"] & C^{n}(A) \arrow[r, "b"] \arrow[u, "B"] & C^{n+1}(A) \arrow[r] \arrow[u, "B"] & \cdots \\
\cdots \arrow[r] & C^{n}(A) \arrow[r, "b"] \arrow[u, "B"] & C^{n+1}(A) \arrow[r, "b"] \arrow[u, "B"] & C^{n+2}(A) \arrow[r] \arrow[u, "B"] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}$$

Explicitly, the component $C^{j,k}$ in the j-th column and k-th row of the complex is $C^{j-k}(A)$, where $j,k \in \mathbb{Z}$.[18] This induces and explains the above bigradings on Hochschild and cyclic cohomology.

Given a double complex as above, there is again a standard procedure to form two associated spectral sequences, which under favourable circumstances converge to the cohomology of the *total complex*

$$(\operatorname{Tot} C)^n := \bigoplus_{j+k=n} C^{j,k}, \quad n \in \mathbb{N}$$

with coboundary $b+B$ (in our context).[19] These correspond to two filtrations of $(\operatorname{Tot} C)^\bullet$. Studying these, one finds that the first leads to a spectral sequence collapsing on the second page. The second, filtration by rows, turns out to be more interesting. Explicitly, this is the decreasing filtration

$$(\operatorname{Tot} C)^n \supseteq \cdots F^q(\operatorname{Tot} C)^n \supseteq F^{q+1}(\operatorname{Tot} C)^n \supseteq \cdots$$

defined by

$$F^q(\operatorname{Tot} C)^n := \bigoplus_{\substack{j+k=n \\ k \geq q}} C^{j,k} = \bigoplus_{k \geq q} C^{n-2k}(A) \quad \forall n \in \mathbb{N}.$$

In [6], Connes also rescales the differentials of the double complex to take care of the factor $n(n+1)$ in $SB = n(n+1)b$.[20] Using arguments similar to those in Theorem 1.30, he then shows:

[18] Using $j-k$ rather than $j+k$ makes it so that B increases k and thus allows working in the cohomological framework.

[19] Note that the double complex we use here is *not* contained in any quadrant. A similar double complex with additional restriction $j,k \geq 0$ is also worth studying (cf. e. g. section 10.1 of [1]) but has different properties.

[20] This is mentioned here only to correctly state the result; we later only use the exact couple setting.

Theorem 1.36 ([6], III.1, Theorem 29d)**.** *The spectral sequence $(E_r, d_r)_r$ associated to the exact couple $(HH^\bullet(A), HC^\bullet(A), S, I, B)$ coincides with the spectral sequence associated to filtration by rows as above, and converges to $HP^\bullet(A)$ filtered by cocycle dimension. Thus,*

$$HC^{ev}(A) \cong \bigoplus_{j\in\mathbb{Z}} E_\infty^{j,-j} \quad \text{and} \quad HC^{odd}(A) \cong \bigoplus_{j\in\mathbb{Z}} E_\infty^{j,1-j}.$$

1.4 Non-commutative vector bundles

The aim of this section is to implement vector bundles and some surrounding notions in the non-commutative setting. After discussing the appropriate generalisation of vector bundles, we describe analogues of the Chern character and connections. The (standard) material of the first subsection can be found in [9] and [1]; the second and third subsections follow section III.3 of [6] and section 4.1 of [9].

1.4.1 Describing vector bundles

In order to generalise vector bundles into our setting, one first needs to describe them by algebraic data. This is accomplished by the Serre-Swan theorem:

Theorem 1.37 (Serre-Swan [18])**.** *Let X be a compact Hausdorff space. For any vector bundle E on X, the $C(X)$-module of continuous sections $\Gamma(E)$ is finitely generated and projective. Conversely, every finitely generated projective module over $C(X)$ is isomorphic to the module of continuous sections of some vector bundle on X.*

Remark 1.38.

(i) The theorem holds for both real and complex vector bundles (with $C(X,\mathbb{R})$ and $C(X,\mathbb{C})$, respectively) though we will only work with the latter.

(ii) There is also a version of the theorem for smooth vector bundles, which states that smooth vector bundles over a smooth manifold X correspond to finitely generated projective modules over $C^\infty(X)$; see chapter 11 of [14].

We recall that finitely generated projective modules over an algebra A can be described in terms of idempotents:

Definition & Proposition 1.39. Let A be a unital algebra.

(i) For $n \in \mathbb{N}$ define $M_n(A)$ to be the algebra of $n \times n$ square matrices over A, with unit group denoted $GL_n(A) \subset M_n(A)$. Moreover, let $M_\infty(A)$ denote the direct limit of the $M_n(A)$ with respect to the inclusions $M_n(A) \hookrightarrow M_{n+1}(A)$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $GL_\infty(A)$ the analogous direct limit of the $GL_n(A)$ with inclusions $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(ii) Associated to any $a \in M_n(A)$, $n \in \mathbb{N}$ there is a map of right modules $A^n \to A^n$ given by (left) matrix multiplication and also denoted a. If $e \in M_n(A)$ is idempotent (i. e. $e^2 = e$), then the image $\operatorname{im} e \subseteq A^n$ is a finitely generated projective

A-module. Conversely, every isomorphism class of finitely generated projective right A-modules arises from an idempotent in this way.

(iii) We denote the set of idempotents in $M_\infty(A)$ by $\mathrm{Idem}(A)$. Idempotents $e_1, e_2 \in \mathrm{Idem}(A)$ are *stably equivalent*, denoted $e_1 \sim e_2$, if there exists $a \in GL_\infty(A)$ such that $e_2 = ae_1a^{-1}$. The modules $\operatorname{im} e_1$ and $\operatorname{im} e_2$ are isomorphic if and only if e_1 and e_2 are stably equivalent.

Proof. We prove the claims made in (ii) and (iii). Both use the idempotent $e^\perp := \mathbb{I} - e$ associated to an idempotent $e \in M_n(A)$ (where $\mathbb{I}$ denotes the identity matrix).

(ii) For an idempotent $e \in M_n(A)$, one checks that $A^n = \operatorname{im} e + \operatorname{im} e^\perp$ is a direct sum, so $\operatorname{im} e$ is projective. Moreover, it is clearly generated by the columns of the matrix e. Conversely, if P is a finitely generated projective A-module, there exist another A-module Q and a natural number n such that $P \oplus Q \cong A^n$. We may identify P and Q with their images in A^n and define $\pi\colon A^n \to A^n$ to be the projection $P \oplus Q \to P$ followed by inclusion into A^n. It is represented by a matrix $e \in M_n(A)$ which is idempotent and has image (isomorphic to) P by construction.

(iii) If $e_2 = ae_1a^{-1}$, then a restricts to an isomorphism of the images. Conversely, suppose $\operatorname{im} e_1 \cong \operatorname{im} e_2$. Choose $n \in \mathbb{N}$ such that e_1 and e_2 have representatives in $M_n(A)$ (which we also denote e_1 and e_2, respectively), and an isomorphism $\phi\colon \operatorname{im} e_1 \to \operatorname{im} e_2$. Considering the endomorphism $A^n \twoheadrightarrow \operatorname{im} e_1 \xrightarrow{\phi} \operatorname{im} e_2 \hookrightarrow A^n$, we can realise ϕ as multiplication with a matrix $f \in M_n(A)$, which then fulfils $e_2 f = f = f e_1$. Similarly, we obtain a matrix $f' \in M_n(A)$ from ϕ^{-1}, with $e_1 f' = f' = f' e_2$ as well as $f'f = e_1$ and $ff' = e_2$. Now observe that in $M_{2n}(A)$ we have

$$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

and further

$$\begin{pmatrix} e_2^\perp & f \\ -f' & e_1^\perp \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix} = \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_2^\perp & f \\ -f' & e_1^\perp \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} e_2^\perp & f \\ -f' & e_1^\perp \end{pmatrix} \begin{pmatrix} e_2^\perp & -f \\ f' & e_1^\perp \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} e_2^\perp & -f \\ f' & e_1^\perp \end{pmatrix} \begin{pmatrix} e_2^\perp & f \\ -f' & e_1^\perp \end{pmatrix},$$

so $\left(\begin{smallmatrix} e_2^\perp & f \\ -f' & e_1^\perp \end{smallmatrix}\right) \in GL_{2n}(A)$ and altogether e_1 and e_2 are stably equivalent.

□

Theorem 1.37 and Proposition 1.39 suggest the following:

Idea 1.40. Let A be a unital but not necessarily commutative algebra. By a *vector bundle* on A we mean a finitely generated projective right A-module or, equivalently, an idempotent of $M_\infty(A)$.

Both vector bundles and finitely generated projective modules become commutative monoids when equipped with the direct sum. In the picture of idempotents, this is realised as follows:

Definition & Proposition 1.41. Let A be a unital algebra.

(i) The operation $\oplus\colon \operatorname{Idem}(A) \times \operatorname{Idem}(A) \to \operatorname{Idem}(A)$ defined on representatives as $(e, f) \mapsto \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$ makes $\operatorname{Idem}(A)$ a commutative monoid.

(ii) For $e_1, e_2, f_1, f_2 \in \operatorname{Idem}(A)$ with $e_1 \sim e_2$ and $f_1 \sim f_2$, we also have $(e_1 \oplus f_1) \sim (e_2 \oplus f_2)$. Thus $\oplus$ descends to the quotient $\operatorname{Idem}(A)/\sim$, which is then also a commutative monoid. We denote the Grothendieck group of $\operatorname{Idem}(A)/\sim$ by $K_0(A)$.

Proof. Part (i) is mostly straightforward up to the technicality of "moving idempotents past zero blocks", e. g. $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right)$, which can be accomplished as in the proof of Proposition 1.39(iii). The neutral element is the class of $0 \in M_1(A)$. For part (ii), one simply combines invertible matrices witnessing the equivalences into a block matrix. □

1.4.2 Action of cyclic cohomology

In this subsection we discuss a pairing between K_0 and even cyclic cohomology. It can be viewed as an analogy of the Chern character for complex vector bundles [6]. In this regard, note that cyclic *cohomology* corresponds to de Rham *homology* (rather than cohomology), so it makes sense for it to be in a pairing with K-theory. For the remainder of this section, A will denote a unital locally convex algebra (as in section 1.3).

We will need a way of constructing cyclic cocycles on matrix algebras $M_k(A)$ from cyclic cocycles on A. To this end, note that $M_k(A) \cong A \otimes M_k(\mathbb{C})$ for any $k \in \mathbb{N}$. On $M_k(\mathbb{C})$, the usual trace of matrices defines a cyclic 0-cocycle $\operatorname{tr} \in Z^0(M_k(\mathbb{C}))$. Thus according to Proposition 1.29, for any $c \in Z^n_\lambda(A)$, $n \in \mathbb{N}$, $c\#\operatorname{tr}$ is a cyclic n-cocycle on $M_k(A)$. In this instance it is not too hard to describe the cup product explicitly and one finds

$$(c\#\operatorname{tr})(a_0 \otimes m_0, \ldots, a_n \otimes m_n) = c(a_0, \ldots, a_n) \operatorname{tr}(m_0 \cdots m_n) \tag{1.21}$$

for all $a_0, \ldots, a_n \in A, m_0, \ldots, m_n \in M_k(\mathbb{C})$.[21]

Equipped with a way of lifting cocycles to $M_\infty(A)$, we now apply them to idempotents. Given an even-dimensional cyclic cocycle $c \in Z^{2n}_\lambda$ (for some $n \in \mathbb{N}$) and an idempotent $e \in \operatorname{Idem}(A)$, we define

$$\langle e, c\rangle := \tfrac{1}{n!}(c\#\operatorname{tr})(e, e, \ldots, e) \in \mathbb{C}, \tag{1.22}$$

where we have identified e with a representative in some $M_k(A)$. Note that (by formula (1.21)) it does not matter which representative we choose, since the embeddings $M_k(A) \hookrightarrow M_{k+1}(A)$ preserve the trace. In fact, the construction is invariant under several further changes:

[21] One could in fact avoid using the cup product entirely by directly verifying that (1.21) defines a cyclic cocycle.

Proposition 1.42. *Let $n, k \in \mathbb{N}$, $c \in Z_\lambda^{2n}(A)$ and $e \in \mathrm{Idem}(A)$, identified with a representative $e \in M_k(A)$.*

(i) $\langle e, \cdot \rangle$ is a linear map $Z_\lambda^{2n}(A) \to \mathbb{C}$.

(ii) If c is a coboundary, i. e. $[c] = 0$ in $HC^{2n}(A)$, then $\langle e, c\rangle = 0$.

(iii) We have $\langle e, S(c)\rangle = \langle e, c\rangle$.

(iv) $\langle \cdot, c\rangle$ is an additive map $\mathrm{Idem}(A) \to \mathbb{C}$.

(v) If $e \sim f$ for $f \in \mathrm{Idem}(A)$, then $\langle e, c\rangle = \langle f, c\rangle$.

Proof.

(i) This follows directly from bilinearity of the cup product (see Proposition 1.29).

(ii) In this case we also have $[c \# \operatorname{tr}] = 0$ in $HC^{2n}(M_k(A))$, cf. Proposition 1.29(iii). Hence $c \# \operatorname{tr} = b\tilde{c}$ for some $\tilde{c} \in Z_\lambda^{2n-1}(M_k(A))$, but then, using idempotence of e,

$$n!\langle e, c\rangle = b\tilde{c}(e, e, \ldots, e) = \sum_{j=0}^{2n} (-1)^j \tilde{c}(e, \ldots, e) = \tilde{c}(e, \ldots, e),$$

and by cyclicity we have $\tilde{c}(e, \ldots, e) = (-1)^{2n-1}\tilde{c}(e, \ldots, e) = 0$.

(iii) First of all, note that we have $S(c) \# \operatorname{tr} = S(c \# \operatorname{tr})$ (which one can check from the definitions). Thus, replacing A by $M_k(A)$, we may assume $k = 1$. Under this assumption we have $e \in A$, and formula (1.21) yields

$$\langle e, c\rangle = \tfrac{1}{n!} c(e, \ldots, e).$$

Now according to (1.20) we have $S(c) = (2n+1)(2n+2)b(\tilde{c})$ for some $\tilde{c} \in C^{2n+1}(A)$ such that $c = B(\tilde{c})$. Then as in the proof of (ii),

$$\langle e, S(c)\rangle = \tfrac{(2n+1)(2n+2)}{(n+1)!} b(\tilde{c})(e, \ldots, e) = \tfrac{2(2n+1)}{n!} \tilde{c}(e, \ldots, e).$$

To compare the results we use $B(\tilde{c}) = c$. Recalling $B = P_\lambda \circ u \circ (1 - \lambda)$ (cf. (1.9)), we compute

$$\begin{aligned}
c(e, \ldots, e) &= (P_\lambda \circ u \circ (1 - \lambda))(\tilde{c})(e, \ldots, e) \\
&= \sum_{j=0}^{2n} (-1)^{2jn} (u \circ (1 - \lambda))(\tilde{c})(e, \ldots, e) \\
&= (2n + 1)(u - u \circ \lambda)(\tilde{c})(e, \ldots, e) \\
&= (2n + 1)(\tilde{c}(1, e, \ldots, e) + \tilde{c}(e, \ldots, e, 1)) \\
&= (2n + 1)\big(2\tilde{c}(e, \ldots, e) + \tilde{c}(e^\perp, e, \ldots, e) + \tilde{c}(e, \ldots, e, e^\perp)\big).
\end{aligned}$$

To finish the proof of (iii) we show $\tilde{c}(e^\perp, e, \dots, e) + \tilde{c}(e, \dots, e, e^\perp) = 0$ using the fact that $b(\tilde{c})$ is even-dimensional and cyclic: on the one hand

$$\begin{aligned} b(\tilde{c})(1, e, \dots, e, e^\perp) &= \tilde{c}(e, \dots, e, e^\perp) + \sum_{j=1}^{2n} (-1)^j \tilde{c}(1, e, \dots, e, e^\perp) \\ &- \tilde{c}(1, e, \dots, e, ee^\perp) + \tilde{c}(e^\perp, e, \dots, e) = \tilde{c}(e^\perp, e, \dots, e) + \tilde{c}(e, \dots, e, e^\perp); \end{aligned}$$

on the other hand

$$\begin{aligned} b(\tilde{c})(e^\perp, 1, e, \dots, e) &= \tilde{c}(e^\perp, e, \dots, e) - \tilde{c}(e^\perp, e, \dots, e) \\ &+ \sum_{j=2}^{2n+1} (-1)^j \tilde{c}(e^\perp, 1, e, \dots, e) + \tilde{c}(ee^\perp, 1, e, \dots, e) = 0. \end{aligned}$$

(iv) This follows from formula (1.21) and the fact that the trace of a block diagonal matrix is the sum of the traces of its blocks.

(v) Let $a \in GL_k(A)$ such that $f = aea^{-1}$. It is sufficient to show that the inner automorphism $x \mapsto axa^{-1}$ of $M_k(A)$ induces the identity map on $HC^\bullet(M_k(A))$. In fact, this is true for inner automorphisms of any locally convex algebra (cf. [6], section III.1, Proposition 8).

□

The following is a direct consequence of Proposition 1.42:

Corollary 1.43. *The assignment*

$$([e], [c]) \mapsto \langle e, c \rangle$$

defines a bilinear map $K_0(A) \times HC^{ev}(A) \to \mathbb{C}$.

1.4.3 Connections and a Chern-Weil approach

Chern classes of complex vector bundles over a smooth manifold can be constructed using connections and their curvature forms in an approach known as Chern-Weil theory. In this subsection we sketch Connes' generalisation of this approach for the pairing described in subsection 1.4.2. A good reference for the classical counterpart is Appendix C of [12].

In classical geometry, a connection on a vector bundle E over a smooth manifold X can be defined as a linear map

$$\nabla \colon \Gamma(E) \to \Gamma(E) \otimes_{C^\infty(X)} \Omega^1(X)$$

such that for all $f \in C^\infty(X)$ and $s \in \Gamma(E)$ we have

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

Moreover, ∇ canonically induces (or extends to) connections on several vector bundles related to E, in particular the bundle $\Omega^*(X,E) \cong E \otimes_{C^\infty(X)} \Omega^*(X)$ of differential forms with values in E.

Let us now generalise this to the non-commutative setting:

Definition 1.44. Fix a unital locally convex dg algebra (Ω, d) with a continuous algebra homomorphism $\rho\colon A \to \Omega^0$, and a finitely generated projective right A-module P. A *connection* on P is a linear map

$$\nabla\colon P \to P \otimes_A \Omega^1$$

such that

$$\nabla(sa) = \nabla(s)a + s \otimes d\rho(a) \quad \forall s \in P, a \in A.$$

Here the A-bimodule structure on Ω^1 is the one induced by ρ.

The extension to bundle-valued differential forms takes the following form:

Proposition 1.45. *For (Ω, d), P, ρ and ∇ as in Definition 1.44, there exists a unique linear endomorphism*

$$\hat{\nabla}\colon P \otimes_A \Omega \to P \otimes_A \Omega$$

such that $\hat{\nabla}(s \otimes \omega) = \nabla(s)\omega + s \otimes d\omega$ for all $s \in P$ and $\omega \in \Omega$. Moreover, for $\tilde{s} \in P \otimes_A \Omega^k$ and $\sigma \in \Omega$ we have

$$\hat{\nabla}(\tilde{s}\sigma) = \hat{\nabla}(\tilde{s})\sigma + (-1)^k \tilde{s} d\sigma \tag{1.23}$$

(where we are using the canonical right Ω-module structure on $P \otimes_A \Omega$).

Proof. The map

$$F\colon P \times \Omega \to P \otimes_A \Omega, \quad (s, \omega) \mapsto \nabla(s)\omega + s \otimes d\omega$$

is $\mathbb{C}$-bilinear and fulfils

$$\begin{aligned} F(s \cdot a, \omega) &= (\nabla(s)a + s \otimes d\rho(a))\omega + sa \otimes d\omega \\ &= \nabla(s)\rho(a)\omega + s \otimes d\rho(a)\omega + s \otimes \rho(a)d\omega \\ &= \nabla(s)\rho(a)\omega + s \otimes d(\rho(a)\omega) \\ &= F(s, a \cdot \omega), \end{aligned}$$

so it induces a linear map $P \otimes_A \Omega \to P \otimes_A \Omega$ which is precisely the desired map $\hat{\nabla}$. Uniqueness is clear.

To check (1.23) we may assume $\tilde{s} = s \otimes \omega$ for $s \in P$ and $\omega \in \Omega^k$. Then

$$\begin{aligned} \hat{\nabla}(\tilde{s}\omega) &= \nabla(s) \otimes (\omega\sigma) + s \otimes d(\omega\sigma) \\ &= (\nabla(s) \otimes \omega + s \otimes d\omega)\sigma + (-1)^k (s \otimes \omega) d\sigma \\ &= \hat{\nabla}(\tilde{s})\sigma + (-1)^k \tilde{s} d\sigma \end{aligned}$$

as claimed. □

While $\hat{\nabla}$ is not a morphism of right Ω-modules due to (1.23), we claim that its square $\hat{\nabla}^2 \equiv \hat{\nabla} \circ \hat{\nabla}$ is. Since $\hat{\nabla}^2$ is $\mathbb{C}$-linear (and in particular additive), it is sufficient to check that for $s \in P$, $\omega \in \Omega^k$ and $\sigma \in \Omega^\ell$ we have

$$\begin{aligned}\hat{\nabla}^2((s \otimes \omega)\sigma) &= \hat{\nabla}(\hat{\nabla}(s \otimes \omega)\sigma + (-1)^k s \otimes \omega d\sigma) \\ &= \hat{\nabla}^2(s \otimes \omega)\sigma + (-1)^{k+1}\hat{\nabla}(s \otimes \omega)d\sigma \\ &+ (-1)^k(\hat{\nabla}(s \otimes \omega)d\sigma + (-1)^k s \otimes \omega d^2\sigma) \\ &= \hat{\nabla}^2(s \otimes \omega)\sigma,\end{aligned}$$

where we have used that $\hat{\nabla}$ maps $P \otimes_A \Omega^k$ into $P \otimes_A \Omega^{k+1}$.

Definition 1.46. With the same context as in Proposition 1.45, we define the *curvature form* associated to the connection ∇ to be $R \coloneqq \hat{\nabla}^2 \in \mathrm{End}_\Omega(P \otimes_A \Omega)$.

To connect this to the pairing from the previous subsection, consider now a cycle $(\Omega, \int, \rho)$ over A of dimension $2n$, $n \in \mathbb{N}$, as in Definition 1.14. By definition, $\int$ is a linear map $\Omega \to \mathbb{C}$ (zero outside Ω^{2n}). It extends[22] to a linear map $\int\colon \mathrm{End}_\Omega(P \otimes_A \Omega)$ as follows: firstly, since P is finitely generated projective over A, $P \otimes_A \Omega$ is finitely generated projective over Ω. Hence $P \otimes_A \Omega$ embeds as a direct summand into $\Omega^{\times N}$ for some $N \in \mathbb{N}$, and this realises $\mathrm{End}_\Omega(P \otimes_A \Omega)$ as a subalgebra of $M_N(\Omega)$. Now on $M_N(\Omega)$ the usual trace of matrices provides a map $\mathrm{tr}_\Omega\colon M_N(\Omega) \to \Omega$. Composing this with $\int$ yields the desired extension $\int\colon \mathrm{End}_\Omega(P \otimes A) \to \mathbb{C}$. With this in place, the relation to the pairing of subsection 1.4.2 is as follows:

Theorem 1.47 ([6], Proposition III.3.8)**.** *Let $n \in \mathbb{N}$, and $(\Omega, \int, \rho)$ be a $2n$-dimensional cycle over A with character $\chi \in Z^{2n}_\lambda(A)$. Given a finitely generated projective A-module P with connection ∇ and curvature form R as above, let $[P] \in K_0(A)$ denote the stable equivalence class of idempotents corresponding to the isomorphism class of P. Then we have*

$$\langle [P], [\chi] \rangle = \frac{1}{n!} \int R^n.$$

Example 1.48. Let X be a $2n$-dimensional compact orientable manifold, and E a complex rank n vector bundle on X with a connection ∇. Then R is precisely the usual curvature form of ∇. Now consider the $2n$-dimensional cycle given by integration on the exterior algebra $\Omega^*(X)$ as in (1.3). Then $\Gamma(E) \otimes_{C^\infty(X)} \Omega^*(X)$ can be identified with the space $\Omega^*(X, E)$ of E-valued differential forms, and $\mathrm{End}_{\Omega^*(X)}(\Omega^*(X, E))$ with $\Omega^*(X, \mathrm{End}(E))$, where $\mathrm{End}(E)$ denotes the bundle endomorphisms of E. Moreover, the extension of $\int$ as constructed above is just integration of the pointwise trace. Hence

$$\frac{1}{n!} \int R^n = \int_X \mathrm{tr}\left(\frac{R^n}{n!}\right),$$

the integral (or evaluation on the fundamental class) of the top component of the Chern character $\mathrm{ch}(E)$. [9]

[22] In a loose sense; here Ω is identified with $\{\mathrm{id}_P\} \otimes \Omega \subseteq \mathrm{End}_A(P) \otimes \Omega \cong \mathrm{End}_\Omega(P \otimes_A \Omega)$, and the extension as constructed below is up to a multiple.

2 Non-commutative Tori

Non-commutative tori are non-commutative spaces in the sense of section 1.2, so their description is in terms of C*-algebras. These C*-algebras $\mathcal{A}_\theta$, called rotation algebras, depend on a parameter $\theta \in [0, 1)$, and $\mathcal{A}_0$ is isomorphic to $C(\mathbb{T}^2)$. Thus, one can view non-commutative tori as deformations of the classical torus $\mathbb{T}^2$, and for this reason the first section of this chapter is devoted to the C*-algebra $C(\mathbb{T}^2)$.

The C*-algebras $\mathcal{A}_\theta$ are among the simplest examples of non-commutative C*-algebras. Moreover, they have several (related) geometric interpretations, which we briefly discuss at the end of section 2.3.

2.1 The commutative case

Our definition of non-commutative tori will be a presentation, i. e. a description in terms of generators and relations. To motivate it we first derive such a description of $C(\mathbb{T}^2)$, the C*-algebra associated to the usual (commutative) 2-torus. We then define non-commutative tori by introducing an additional, non-commutative relation.

It is convenient here to realise the torus as $S^1 \times S^1$ and view S^1 as the unit circle in $\mathbb{C}$. With these conventions, consider the functions $U, V \in C(\mathbb{T}^2)$ defined by

$$U(z, w) \coloneqq z, \quad V(z, w) \coloneqq w \quad \forall (z, w) \in S^1 \times S^1.$$

Clearly $U^*U = 1 = V^*V$, so both U and V are unitary elements of $C(\mathbb{T}^2)$. The subalgebra generated by U, V, U^{-1}, V^{-1} is dense by the Stone-Weierstrass theorem. Moreover, this subalgebra is certainly contained in the C*-subalgebra generated by U and V, so we have $C(\mathbb{T}^2) = C^*(U, V)$.

U and V fulfil no obvious relations, so one might conjecture that $C(\mathbb{T}^2)$ is *freely* generated, as a commutative C*-algebra, by U and V. This is essentially true, but the word ‘free’ is not used in this context. The category of C*-algebras is too restrictive to allow for free objects analogous to free groups[1], so not every set of generators and relations can be realised as a C*-algebra. When this *is* possible, one speaks of *universal* C*-algebras:

[1]For example, there is no free C*-algebra on one generator. If it existed, it would be given by a C*-algebra A with generator a such that for every C*-algebra B and every $b \in B$ there is a unique morphism $\phi\colon A \to B$ with $\phi(a) = b$. As morphisms are norm-decreasing, this yields $\|a\| \leq \|b\|$. Since b was arbitrary, we get $a = 0$, which is impossible. Observe that this particular problem can be avoided by imposing restrictions on the norm of b as in the case of unitary generators.

Proposition 2.1. *$C(\mathbb{T}^2)$ is the*[2] *universal C*-algebra on two commuting unitary generators. That is, given any unital C*-algebra A and any pair of commuting unitary elements $\tilde{U}, \tilde{V}$, there is a unique morphism $\phi\colon C(\mathbb{T}^2) \to A$ such that $\phi(U) = \tilde{U}$ and $\phi(V) = \tilde{V}$.*

Proof. Existence of ϕ can be checked directly, or using the Gelfand transform from Theorem 1.8, as follows. By assumption, the C*-subalgebra $B := C^*(\tilde{U}, \tilde{V})$ of A is unital and commutative. We consider the continuous function

$$F\colon G(B) \to \mathbb{T}^2, \quad \mu \mapsto (\mu(\tilde{U}), \mu(\tilde{V})),$$

which is well-defined because unitary elements have spectrum contained in S^1.[3] Applying the functor C from (1.1), we obtain a morphism

$$C(F)\colon C(\mathbb{T}^2) \to C(G(B)), \quad h \mapsto h \circ F.$$

We have $(C(F)(U))(\mu) = \mu(\tilde{U})$ for all characters $\mu \in G(B)$, which we can express as $C(F)(U) = \mathcal{G}(\tilde{U})$, where $\mathcal{G}$ denotes the Gelfand transform. Similarly, $C(F)(V) = \mathcal{G}(\tilde{V})$. Therefore, the morphism $\phi := \mathcal{G}^{-1} \circ C(F)$ has the required properties. Uniqueness is clear as $\{U, V\}$ generates $C(\mathbb{T}^2)$. □

Proposition 2.1 gives a C*-algebraic description of the torus as a *topological space*. In order to describe it as a *smooth manifold*, we should use smooth functions, i. e. $C^\infty(\mathbb{T}^2) \subseteq C(\mathbb{T}^2)$. Here it is convenient to make use of Fourier series, in particular the fact that a smooth function $\mathbb{T}^2 \to \mathbb{C}$ can be represented as a Fourier series $\sum_{k,\ell\in\mathbb{Z}} a_{k\ell} U^k V^\ell$. Moreover, the coefficients $a_{k,\ell}$ of a smooth function are of rapid decay, and this in fact characterises smooth functions on $\mathbb{T}^2$.

More precisely, we define a family of seminorms $\|\cdot\|_n$, where $n \in \mathbb{N}$, on the space of complex-valued sequences $(a_{k,\ell})_{k,\ell\in\mathbb{Z}}$ by

$$\|a\|_n := \sup_{k,\ell\in\mathbb{Z}} (|k| + |\ell|)^n |a_{k,\ell}|, \tag{2.1}$$

and the *Schwartz space* $\mathcal{S}(\mathbb{Z}^2)$ as

$$\mathcal{S}(\mathbb{Z}^2) := \{a\colon \mathbb{Z}^2 \to \mathbb{C} \mid \|a\|_n < \infty \text{ for all } n \in \mathbb{N}\}.$$

Smooth functions $\mathbb{T}^2 \to \mathbb{C}$ are then precisely those whose Fourier coefficients are contained in $\mathcal{S}(\mathbb{Z}^2)$. In summary, we have the following:

Proposition 2.2. *The dense subalgebra $C^\infty(\mathbb{T}^2) \subseteq C(\mathbb{T}^2) = C^*(U, V)$ can be described as*

$$C^\infty(\mathbb{T}^2) = \Big\{ \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} U^k V^\ell \mid (a_{k,\ell})_{k,\ell\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}^2) \Big\}.$$

[2] As with other universal objects, the definite article 'the' only applies up to isomorphism.

[3] This is where the geometry of $\mathbb{T}^2$ is used. The Gelfand spectrum of any C*-algebra generated by two unitaries can be embedded into $\mathbb{T}^2$.

We can take this one step further and work directly in $\mathcal{S}(\mathbb{Z}^2)$ by transferring the relevant structure from $C^\infty(\mathbb{T}^2)$. That is, we make $\mathcal{S}(\mathbb{Z}^2)$ an algebra via the convolution product

$$(a * b)_{m,n} \coloneqq \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} b_{m-k,n-\ell}, \quad a, b \in \mathcal{S}(\mathbb{Z}^2),\ m, n \in \mathbb{Z}. \tag{2.2}$$

As for the topology, $C(\mathbb{T}^2)$ is a Fréchet algebra. By equipping $\mathcal{S}(\mathbb{Z}^2)$ with the seminorms from (2.1), it becomes a Fréchet algebra as well, and we have:

Proposition 2.3. *The Fréchet algebra* $\mathbb{C}^\infty(\mathbb{T}^2)$ *is isomorphic to* $(\mathcal{S}(\mathbb{Z}^2), *)$.

2.2 Definition and first properties

Here we define the C*-algebras corresponding to non-commutative tori, and discuss their dense subalgebras modelling smooth functions. Afterwards we construct a faithful trace on them, (partially) following [8].

Starting from Proposition 2.1, we may introduce non-commutativity in the form of a relation between the generators U and V.

Definition 2.4. Let $\theta \in [0, 1)$. We define $\mathcal{A}_\theta$ to be the universal C*-algebra generated by two unitary elements U, V, subject to the relation

$$VU = e^{2\pi i\theta} UV. \tag{2.3}$$

By universality, we mean that for any unital C*-algebra A and any unitary elements $\tilde{U}, \tilde{V} \in B$ with $\tilde{V}\tilde{U} = e^{2\pi i\theta}\tilde{U}\tilde{V}$, there exists a unique morphism $\phi\colon \mathcal{A}_\theta \to A$ such that $\phi(U) = \tilde{U}$ and $\phi(V) = \tilde{V}$. We occasionally shorten notation to $e^{2\pi i\theta} =: q$, so that $VU = qUV$.

Universality defines $\mathcal{A}_\theta$ uniquely up to isomorphism, but we have not yet shown that it exists at all. An explicit construction is described in section 2.3.

By construction (and Proposition 2.1), $\mathcal{A}_\theta$ is similar to $C(\mathbb{T}^2)$, and even isomorphic to it for $\theta = 0$. Therefore, we may think of $\mathcal{A}_\theta$ as the C*-algebra associated to a *non-commutative torus* $\mathbb{T}^2_\theta$. The algebras $\mathcal{A}_\theta$ are often called *rotation algebras*[4], and sometimes also themselves referred to as non-commutative tori.

Consider now a rapidly decaying sequence $a \in \mathcal{S}(\mathbb{Z}^2)$. Since U and V are unitary, we have the estimate

$$\Big\| \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} U^k V^\ell \Big\| \leq \sum_{k,\ell\in\mathbb{Z}} |a_{k,\ell}| < \infty,$$

so the series $\sum_{k,\ell} a_{k,\ell} U^k V^\ell$ converges to some element of $\mathcal{A}_\theta$. This yields a dense subspace

$$\tilde{\mathcal{A}}_\theta \coloneqq \Big\{ \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} U^k V^\ell \mid a \in \mathcal{S}(\mathbb{Z}^2) \Big\} \subseteq \mathcal{A}_\theta.$$

[4] An explanation of the name can be found in the next section.

The product of two series of the above form is given by

$$\Big(\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell\Big)\Big(\sum_{k,\ell\in\mathbb{Z}} b_{k',\ell'}U^{k'}V^{\ell'}\Big) = \sum_{m,n\in\mathbb{Z}}\sum_{\substack{k+k'=m\\ \ell+\ell'=n}} q^{\ell k'}a_{k,\ell}b_{k',\ell'}U^mV^n.$$

To describe this via the coefficient sequences, we define

$$(a *_\theta b)_{m,n} \coloneqq \sum_{k,\ell\in\mathbb{Z}} e^{2\pi i\ell(m-k)\theta}a_{k,\ell}b_{m-k,n-\ell}, \quad a,b\in\mathcal{S}(\mathbb{Z}^2),\ m,n\in\mathbb{Z}, \tag{2.4}$$

which can be viewed as a modified convolution. Observe that we have the estimate

$$|(a *_\theta b)_{m,n}| \le \sum_{k,\ell\in\mathbb{Z}} |a_{k,\ell}||b_{m-k,n-\ell}|.$$

Since for $a,b\in\mathcal{S}(\mathbb{Z}^2)$ also the sequences $|a|,|b|$, obtained by taking the absolute value of each entry, are in $\mathcal{S}(\mathbb{Z}^2)$, the right-hand side is the (m,n) entry of $|a| * |b| \in \mathcal{S}(\mathbb{Z}^2)$. From this we conclude that $a *_\theta b \in \mathcal{S}(\mathbb{Z}^2)$, so $\mathcal{S}(\mathbb{Z}^2)$ is closed under $*_\theta$ and thus $\tilde{\mathcal{A}}_\theta$ is a subalgebra of $\mathcal{A}_\theta$.

Remark 2.5. In light of Proposition 2.2, one should interpret $\tilde{\mathcal{A}}_\theta$ as the algebra of smooth functions on the non-commutative torus $\mathbb{T}^2_\theta$. As in Proposition 2.3, mapping sequences to power series in U and V yields an isomorphism of Fréchet algebras $\tilde{\mathcal{A}}_\theta \cong (\mathcal{S}(\mathbb{Z}^2), *_\theta)$ (with the same Fréchet space structure on $\mathcal{S}(\mathbb{Z}^2)$).

It is useful to also describe the involution in the Schwartz space picture. Since

$$(a_{k,\ell}U^kV^\ell)^* = a^*_{k,\ell}V^{-\ell}U^{-k} = a^*_{k,\ell}q^{k\ell}U^{-k}V^{-\ell},$$

the correct definition is

$$(a^*)_{k,\ell} \coloneqq e^{2\pi ik\ell\theta}(a_{-k,-\ell})^*, \quad a\in\mathcal{S}(\mathbb{Z}^2). \tag{2.5}$$

In summary:

Proposition 2.6. *The map*

$$\phi\colon \mathcal{S}(\mathbb{Z}^2)\to\mathcal{A}_\theta, \quad a\mapsto \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell$$

is a continuous algebra homomorphism fulfilling $\phi(a^*) = \phi(a)^*$ for all $a\in\mathcal{S}(\mathbb{Z}^2)$. Its image is the dense subalgebra $\tilde{\mathcal{A}}_\theta \subseteq \mathcal{A}_\theta$.

Using $\tilde{\mathcal{A}}_\theta$, we can construct a trace τ on $\mathcal{A}_\theta$ as follows. On $\tilde{\mathcal{A}}_\theta$, define

$$\tilde\tau\Big(\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell\Big) \coloneqq a_{0,0}\in\mathbb{C}.$$

This is clearly a bounded linear map. In the equivalent picture of $\mathcal{S}(\mathbb{Z}^2)$ (using Proposition 2.6), it corresponds to the projection to the $0,0$ entry.

Next, given $a, b \in \mathcal{S}(\mathbb{Z}^2)$, we compute

$$(a *_\theta b)_{0,0} = \sum_{k,\ell\in\mathbb{Z}} q^{-k\ell} a_{k,\ell} b_{-k,-\ell}, \tag{2.6}$$

which is symmetric in a and b. It follows that $\tilde{\tau}(xy) = \tilde{\tau}(yx)$ for all $x, y \in \tilde{\mathcal{A}}_\theta$.

Similarly, given $a \in \mathcal{S}(\mathbb{Z}^2)$ we can simplify (2.6) to

$$(a^* *_\theta a)_{0,0} = \sum_{k,\ell\in\mathbb{Z}} q^{-k\ell} a_{k,\ell} q^{k\ell} a^*_{k,\ell} = \sum_{k,\ell\in\mathbb{Z}} |a_{k,\ell}|^2,$$

which shows that $\phi(x^*x) \geq 0$ for all $x \in \tilde{\mathcal{A}}_\theta$, with equality only for $x = 0$.

We now define $\tau\colon \mathcal{A}_\theta \to \mathbb{C}$ as the unique bounded linear extension of $\tilde{\tau}$. The properties of $\tilde{\tau}$ derived above then extend to τ, so we have:

Proposition 2.7. *The map τ defined above is a faithful trace on $\mathcal{A}_\theta$.*

We remark that τ can also be constructed via averages of an action of $\mathbb{T}^2 = U(1)\times U(1)$ on $\mathcal{A}_\theta$, as done e. g. in [8, p. 168].

If θ is irrational, τ admits a technically useful expression in terms of conjugations with the generators. To that end, first observe that for $k, \ell, m \in \mathbb{Z}$ we have

$$U^{-m}(U^k V^\ell)U^m = e^{2\pi i m\ell\theta} U^k V^\ell.$$

Averaging over $-M \leq m \leq M$ for $M \in \mathbb{N}$, we get

$$\frac{1}{2M+1} \sum_{m=-M}^{M} U^{-m}(U^k V^\ell)U^m = D_M(\ell\theta) U^k V^\ell, \tag{2.7}$$

where

$$D_M(t) \coloneqq \frac{1}{2M+1} \sum_{m=-M}^{M} e^{2\pi i m t}, \quad t \in \mathbb{R}$$

denotes the *Dirichlet kernel* known from Fourier analysis (normalised to $D_M(0) = 1$).

What follows is again clearer in the picture of $\mathcal{S}(\mathbb{Z}^2)$, where we denote the elements corresponding to U, V by u, v.[5] Moreover, we suppress the multiplication symbol $*_\theta$ for cleaner notation. Equation (2.7) then also holds for lower-case u and v. By continuity, (2.7) generalises from the monomials $u^k v^\ell$ to arbitrary elements of $\mathcal{S}(\mathbb{Z}^2)$, yielding

$$\left(\frac{1}{2M+1} \sum_{m=-M}^{M} u^{-m} a u^m\right)_{k,\ell} = D_M(\ell\theta) a_{k,\ell}, \quad a \in \mathcal{S}(\mathbb{Z}^2). \tag{2.8}$$

We now use the assumption that θ is irrational, which implies that $\ell\theta$ is only an integer for $\ell = 0$. Then the explicit formula for the Dirichlet kernel,

$$D_M(t) = \frac{1}{2M+1} \frac{\sin((2M+1)\pi t)}{\sin(\pi t)}, \quad t \in \mathbb{R} \setminus \mathbb{Z},$$

[5] That is, $u_{k,\ell} = \delta_{1k}\delta_{0\ell}$ and $v_{k,\ell} = \delta_{0k}\delta_{1\ell}$.

shows that $D_M(\ell\theta) \to 0$ as $M \to \infty$, for all $\ell \neq 0$. We now take this limit in (2.8). Since D_M is bounded by 1, and the norm on $\mathcal{A}_\theta$ restricted to $\tilde{\mathcal{A}}_\theta$ is controlled by the ℓ^2-norm, we can use dominated convergence to conclude that

$$\left(\lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-M}^{M} u^{-m} a u^m \right)_{k,\ell} = \delta_{\ell 0} a_{k,\ell}.$$

Thus the map $\widetilde{\Phi_1} \colon \tilde{\mathcal{A}}_\theta \to \mathcal{A}_\theta$, defined by

$$\widetilde{\Phi_1}(x) := \lim_{M\to\infty} \frac{1}{2M+1} U^{-m} x U^m, \quad x \in \tilde{\mathcal{A}}_\theta, \tag{2.9}$$

is the projection to the part without V, i. e. to $C^*(U) \cap \tilde{\mathcal{A}}_\theta$. Since this is clearly continuous, we can extend it to $\Phi_1 \colon \mathcal{A}_\theta \to \mathcal{A}_\theta$, and one readily checks that this extension is given by the same formula, i. e.

$$\Phi_1(x) := \lim_{M\to\infty} \frac{1}{2M+1} U^{-m} x U^m, \quad x \in \mathcal{A}_\theta. \tag{2.10}$$

One similarly defines $\Phi_2 \colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ with V in place of U, which on $\tilde{\mathcal{A}}_\theta$ projects to the part without U.[6]

Still assuming $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the above characterises τ by

$$\Phi_2(\Phi_1(x)) = \tau(x)1, \quad x \in \mathcal{A}_\theta. \tag{2.11}$$

Indeed this holds on $\tilde{\mathcal{A}}_\theta$ by the above, and extends by continuity. An application is the following [8]:

Proposition 2.8. *$\mathcal{A}_\theta$ is simple if θ is irrational.*

Proof. Let $I \subset \mathcal{A}_\theta$ be an ideal. Then I is closed under conjugation by U^m for all $m \in \mathbb{Z}$, so the formula (2.10) shows that $\Phi_1(I) \subseteq I$. Similarly, $\Phi_2(I) \subseteq I$, but then (2.11) implies $\tau(I)e \subseteq I$. If $I \neq 0$, it must contain a nonzero positive element x (e. g. y^*y for any $y \in I \setminus \{0\}$). Then $\tau(x) > 0$ by Proposition 2.7, which implies $1 \in I$ and thus $I = \mathcal{A}_\theta$. □

Assuming existence of $\mathcal{A}_\theta$ (see next section), this shows that all realisations of the commutation relation (2.3) are equivalent:

Corollary 2.9. *If θ is irrational, all unital C*-algebras generated by unitary elements U, V fulfilling relation (2.3) are isomorphic (to $\mathcal{A}_\theta$).*

Proof. Let B, with unitaries $\tilde{U}, \tilde{V}$ be such a C*-algebra. Then, by definition of $\mathcal{A}_\theta$, there is a surjective morphism $\phi \colon \mathcal{A}_\theta \to B$. Surjectivity implies that the ideal $\ker\phi$ cannot equal $\mathcal{A}_\theta$, hence the kernel is trivial by simplicity of $\mathcal{A}_\theta$, and ϕ is an isomorphism. □

As will be visible in the next section, there *are* non-isomorphic C*-algebras realising relation (2.3) if θ is rational, leading to the perhaps surprising fact that rational rotation algebras are more complicated to construct than their irrational siblings.

[6] Φ_1 and Φ_2 can also be described more conceptually, as averages of an action of $U(1)$ on $\mathcal{A}_\theta$, see e. g. [8, p. 167].

2.3 Existence and further descriptions

We now describe a way of constructing $\mathcal{A}_\theta$. In the C*-algebra $\mathcal{L}(L^2(S^1))$ of bounded linear operators on $L^2(S^1)$, consider the elements U, V given by

$$(Uf)(z) := zf(z), \quad (Vf)(z) := f(e^{2\pi i\theta}z) \quad \forall f \in L^2(S^1),\, z \in S^1.$$

It is easy to see that U and V are norm-preserving and hence unitary. Moreover, we have

$$UVf(z) = zf(e^{2\pi i\theta}z)$$

and

$$VUf(z) = e^{2\pi i\theta}zf(e^{2\pi i\theta}z) = e^{2\pi i\theta}UVf(z)$$

for all $f \in L^2(S^1)$ and all $z \in S^1$, so these operators satisfy relation (2.3). We claim that for irrational θ the C*-subalgebra $A := C^*(U,V) \subseteq \mathcal{L}(L^2(S^1))$ generated by U and V is universal in the sense of Definition 2.4. If θ is rational, we will see that this is more subtle.

To prove our claim, it is convenient to work with the dense subalgebra of A which, assuming our claim, corresponds to $\tilde{\mathcal{A}}_\theta$. That is, we consider the subset

$$A_0 := \Big\{ \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell \mid (a_{k,\ell}) \in \mathcal{S}(\mathbb{Z}^2)\Big\},$$

which (as in the previous section) is a subalgebra of A. Moreover, A_0 is clearly closed under taking adjoints. By continuity, closure under addition, multiplication, and taking adjoints extends to the (topological) closure $\overline{A_0}$, from which we conclude that $A = \overline{A_0}$.

Passing from series with rapidly decaying coefficients to finite sums, we get the subalgebra A_{00} generated by U, V, U^*, V^*. Since A_{00} is dense in A_0, we conclude from the previous paragraph that $A = \overline{A_{00}}$. We now establish that the monomials U^kV^ℓ form a basis of A_{00}.

Lemma 2.10. *The set $\{U^kV^\ell : k, \ell \in \mathbb{Z}\} \subseteq \mathcal{L}(L^2(S^1))$ is linearly independent and hence a basis of A_{00}. Moreover, the map $\mathbb{Z}^2 \to \mathcal{L}(L^2(S^1))$ given by $(k,\ell) \mapsto U^kV^\ell$ is injective if and only if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof. We use the orthonormal basis $(e_m)_{m\in\mathbb{Z}}$ of $L^2(S^1)$ given by $e_m(z) = z^k$. With respect to this basis, U and V take on the simple forms

$$U(e_m) = e_{m+1}, \quad V(e_m) = q^m e_m \quad \forall m \in \mathbb{Z},$$

so they correspond to a shift operator and a multiplication operator.

Now suppose we have

$$\sum_{k,\ell} a_{k,\ell}U^kV^\ell = 0 \tag{2.12}$$

for a finite linear combination. Applying it to e_m yields

$$\sum_k \sum_\ell a_{k,\ell}\, q^{m\ell} e_{m+k} = 0,$$

which implies

$$\sum_{\ell} a_{k,\ell}\, q^{m\ell} = 0$$

for every (fixed) k. Varying m, we can re-interpret this as

$$\sum_{\ell} a_{k,\ell} e_{\ell}(q^m) = 0 \quad \forall m \in \mathbb{Z}. \tag{2.13}$$

At this point, we distinguish the cases $\theta \notin \mathbb{Q}$ and $\theta \in \mathbb{Q}$.

If θ is irrational, the set $\{q^m : m \in \mathbb{Z}\}$ is infinite. Then $\sum_{\ell} a_{k,\ell} e_{\ell}$ is the restriction to S^1 of a polynomial with infinitely many zeros, hence zero everywhere.

If $\theta = c/d \in \mathbb{Q}$, with $c, d \in \mathbb{Z}$ relatively prime, then $V^{\ell+d} = V^{\ell}$ for all ℓ.[7] Hence, we may assume that ℓ ranges from 0 to $d-1$ in (2.12) and (2.13). Again, the left-hand side is the restriction of a polynomial, which in this case has d distinct zeros (at all the d-th roots of unity q^m, $m = 0, \dots, d-1$). Since the maximal value of ℓ, which is an upper bound for the polynomial's degree, is $d-1$, the polynomial must be zero.

In both cases, we conclude that $a_{k,\ell} = 0$ for all ℓ and all k, which proves linear independence. The second assertion is now clear. □

Now assume $\theta \notin \mathbb{Q}$, and let B be an arbitrary unital C*-algebra B with two unitary elements $\tilde{U}, \tilde{V}$ fulfilling $\tilde{V}\tilde{U} = e^{2\pi i\theta}\tilde{U}\tilde{V}$. Then by Lemma 2.10, we can uniquely define a linear map $\phi_{00} \colon A_{00} \to B$ by setting

$$\phi_{00}(U^k V^{\ell}) := \tilde{U}^k \tilde{V}^{\ell}, \quad k, \ell \in \mathbb{Z}.$$

Since $U, V, \tilde{U}, \tilde{V}$ are unitaries, ϕ_{00} is bounded and can thus be extended to a linear map $\phi \colon A \to B$. Moreover, ϕ_{00} by construction preserves multiplication and the *-involution. By continuity, so does ϕ, so it is a morphism of C*-algebras. We have now verified that A satisfies the definition of $\mathcal{A}_{\theta}$.

Remark 2.11. In light of Corollary 2.9, it is not surprising that the preceding construction realises $\mathcal{A}_{\theta}$. However, the explicit verification was indeed necessary because Corollary 2.9 hinges on the existence of some realisation of $\mathcal{A}_{\theta}$. We could instead have established this existence more abstractly, and immediately concluded $A \cong \mathcal{A}_{\theta}$ from the corollary. See [8] for such an approach.

In case $\theta \in \mathbb{Q}$, expressed in least terms as c/d, the same construction for ϕ only works under the additional assumption that $\tilde{V}^q = 1$. However, it is not hard to find examples of $(B, \tilde{U}, \tilde{V})$ for which this is not the case. For instance, one can modify the C*-algebra A from above by setting

$$\tilde{U} f(z) := z^2 f(z), \quad \tilde{V} f(z) := f(e^{\pi i\theta z}), \quad \forall f \in L^2(S^1), z \in S^1,$$

[7] In this case, writing $\ell \in \mathbb{Z}$, as in the statement of the lemma, may be somewhat misleading, since only finitely many second indices are needed.

and then $\tilde{V}$ has order $2q$ instead of q. Even more drastically, one can take the original construction with $-\theta$ in place of θ and swap U and V. Nevertheless, $\mathcal{A}_\theta$ also exists for $\theta \in \mathbb{Q}$. Explicit constructions can be found e. g. in [2] or the expository paper [10].

That being said, there are several other approaches to and definitions of the C*-algebras $\mathcal{A}_\theta$. In particular, their existence follows from more general constructions. Here we briefly mention some of them, as well as related concepts.

Firstly, $\mathcal{A}_\theta$ can be viewed as a *twisted group C*-algebra* of the group $\mathbb{Z}^2$, cf. [9, p.62]. The group C*-algebra of a locally compact group G is the C*-algebraic analogue of its group algebra $\mathbb{C}[G]$, and is important in its representation theory (on Hilbert spaces). To construct it, one can start with the Banach algebra $L^1(G)$ (defined using Haar measure) equipped with the convolution product. This can be made a *-algebra, which in general does not fulfil the C*-property. From there, one obtains the group C*-algebra loosely speaking by taking an appropriate completion with respect to a certain stronger norm. The twisted group C*-algebra is obtained similarly after deforming the convolution product in $L^1(G)$. For $G = \mathbb{Z}^2$, the usual convolution product on $L^1(\mathbb{Z}^2) = \ell^1(\mathbb{Z}^2)$ is defined as in (2.2), whereas the modified product in (2.4) is an example of such a deformation.

Secondly, $\mathcal{A}_\theta$ is associated with the action of $\mathbb{Z}$ on the circle S^1 by the rotations $m \cdot z := e^{2\pi i m\theta} z$, where $m \in \mathbb{Z}$ and $z \in S^1$. This induces what is also called an *action* of $\mathbb{Z}$ on the C*-algebra $C(S^1)$. More generally, given an action α of a locally compact group G on a C*-algebra A, one can form the so-called *crossed product* C*-algebra $A \rtimes_\alpha G$, cf. [6, p.171]. The rotation algebras $\mathcal{A}_\theta$ are (isomorphic to) the crossed products $C(S^1) \rtimes \mathbb{Z}$ with respect to the action mentioned above. This explains the name 'rotation algebra'. [1, p.524–526].

Another C*-algebra related to $\mathcal{A}_\theta$ is the one associated to the foliation of the torus $\mathbb{T}^2$ by lines of slope θ (in the picture $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$). Again, this is an instance of a more general construction assigning a C*-algebra to a foliated manifold via its holonomy groupoid, as described in [6]. One can show that this foliation C*-algebra is Morita equivalent to $\mathcal{A}_\theta$, see [1, ch. 12] for a discussion. This, as well as the previous example, sheds light on the importance of whether θ is rational. Indeed the action of $\mathbb{Z}$ on S^1 and the foliation of $\mathbb{T}^2$ both have significantly different properties for rational θ.

Lastly, it should be mentioned that (for irrational θ) apart from the particular explicit realisation discussed above, several others appear naturally. By corollary 2.9, any pair of operators U, V satisfying (2.3) generate a realisation of $\mathcal{A}_\theta$. Examples of such pairs of operators include those coming from the time evolution of position and momentum operators in quantum mechanics, as well as the time and frequency shift operators of time-frequency analysis.[8]

2.4 Derivations and basic cyclic cocycles

This section treats two related pieces of structure on non-commutative tori which have close analogues in the commutative case. First we describe two basic derivations corre-

[8] In fact, our realisation can be viewed as an instance of these on S^1.

sponding to the standard coordinate vector fields on $\mathbb{T}^2$. We then use these derivations to construct four explicit cyclic cocycles. In section 2.5 it is shown that these cyclic cocycles generate the periodic cyclic cohomology.

Definition 2.12. A *derivation* on a topological algebra A is a continuous linear map $f\colon A \to A$ such that $f(ab) = f(a)b + af(b)$ for all $a, b \in A$.

To conveniently describe a derivation $f\colon \tilde{A}_\theta \to \tilde{A}_\theta$, we can use that

$$\begin{aligned} f\Big(\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell\Big) &= \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}(f(U^k)V^\ell + U^kf(V^\ell)) \\ &= \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}(U\cdots Uf(U)U\cdots UV^\ell + U^kV\cdots Vf(V)V\cdots V), \end{aligned}$$

so f is completely determined by $f(U)$ and $f(V)$. Two particular choices are δ_1 and δ_2 with

$$\delta_1(U) := 2\pi iU, \quad \delta_1(V) := 0 \tag{2.14}$$

$$\delta_2(U) := 0, \quad \delta_2(V) := 2\pi iV \tag{2.15}$$

These definitions amount to the explicit formulae

$$\delta_1\Big(\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell\Big) = \sum_{k,\ell\in\mathbb{Z}} (2\pi ik)a_{k,\ell}U^kV^\ell, \tag{2.16}$$

$$\delta_2\Big(\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}U^kV^\ell\Big) = \sum_{k,\ell\in\mathbb{Z}} (2\pi i\ell)a_{k,\ell}U^kV^\ell \tag{2.17}$$

for $a \in \mathcal{S}(\mathbb{Z}^2)$, where convergence of the sums as well as continuity of δ_1, δ_2 follows from the definition of $\mathcal{S}(\mathbb{Z}^2)$.

Remark 2.13 (Interpretation as vector fields). In the commutative case $\theta = 0$, δ_1, δ_2 can be viewed as derivations of $C^\infty(\mathbb{T}^2)$ and thus as vector fields on $\mathbb{T}^2$. A convenient picture for this comes from the covering map $p\colon \mathbb{R}^2 \to \mathbb{T}^2$ given by $p(s,t) := (e^{2\pi is}, e^{2\pi it})$. The push-forwards $p_*\partial_1$, $p_*\partial_2$ of the coordinate vector fields on $\mathbb{R}^2$ form a global frame of $\mathbb{T}^2$. To describe their action on a function $f \in C^\infty(\mathbb{T}^2)$, lift f to $\tilde{f} \in C^\infty(\mathbb{R}^2)$, so that

$$(p_*\partial_j)(f)(e^{2\pi is}, e^{2\pi it}) = (\partial_i\tilde{f})(s,t) \quad \forall s,t \in \mathbb{R}, j = 1, 2.$$

But $\tilde{f}$ is precisely the Fourier series of f viewed as a (doubly) periodic function on $\mathbb{R}^2$. It follows that $p_*\partial_j$ is precisely the vector field corresponding to the derivation δ_j.

The trace τ and the derivations δ_1, δ_2 can be combined to define cyclic cocycles on $\tilde{\mathcal{A}}_\theta$ [6][13]. This hinges on the following observations:

Lemma 2.14.

(i) δ_1 and δ_2 commute, that is, $\delta_1(\delta_2(a)) = \delta_2(\delta_1(a))$ for all $a \in \tilde{\mathcal{A}}_\theta$.

(ii) $\tau \circ \delta_1 = \tau \circ \delta_2 = 0$.

Proof. Both assertions follow immediately from (2.16) and (2.17). $\square$

Definition & Proposition 2.15. Define Hochschild cochains $\phi_1, \phi_2 \in C^1(\tilde{\mathcal{A}}_\theta)$ and $\phi_{12} \in C^2(\tilde{\mathcal{A}}_\theta)$ by

$$\phi_j(a_0, a_1) := \tau(a_0 \delta_j(a_1)), \quad j = 1, 2$$

and

$$\phi_{12}(a_0, a_1, a_2) := \tau(a_0 \delta_1(a_1)\delta_2(a_2)) - \tau(a_0 \delta_2(a_1)\delta_1(a_2))$$

for $a_0, a_1, a_2 \in \tilde{\mathcal{A}}_\theta$. Then $\phi_1, \phi_2, \phi_{12}$ are cyclic cocycles.

Proof. Let $a_0, a_1, a_2, a_3 \in \tilde{\mathcal{A}}_\theta$, and $j \in \{1, 2\}$. Firstly, we have

$$\phi_j(a_1, a_0) = \tau(\delta_j(a_0)a_1) = \underbrace{\tau(\delta_j(a_0 a_1))}_{=0} - \tau(a_0 \delta_j(a_1)) = -\phi_j(a_0, a_1),$$

using traciality of τ, the derivation property of δ_j, and Lemma 2.14. This establishes that ϕ_j is cyclic. Similarly,

$$\begin{aligned}
\phi_{12}(a_2, a_0, a_1) &= \tau(\delta_1(a_0)\delta_2(a_1)a_2) - \tau(\delta_2(a_0)\delta_1(a_1)a_2) \\
&= -\tau(a_0\delta_1(\delta_2(a_1))a_2) - \tau(a_0\delta_2(a_1)\delta_1(a_2)) \\
&\quad + \tau(a_0\delta_2(\delta_1(a_1))a_2) + \tau(a_0\delta_1(a_1)\delta_2(a_2)) \\
&= \phi_{12}(a_0, a_1, a_2),
\end{aligned}$$

so ϕ_{12} is cyclic.

Using cyclicity, we have

$$\begin{aligned}
b(\phi_j)(a_0, a_1, a_2) &= \tau(a_0 a_1 \delta_j(a_2)) + \tau(a_1 a_2 \delta_j(a_0)) + \tau(a_2 a_0 \delta_j(a_1)) \\
&= \tau(\delta_j(a_0 a_1 a_2)) = 0.
\end{aligned}$$

Finally, both summands of ϕ_{12} are cocycles individually. For example,

$$\begin{aligned}
&\tau(a_0 a_1 \delta_1(a_2)\delta_2(a_3)) - \tau(a_0\delta_1(a_1 a_2)\delta_2(a_3)) \\
&\quad + \tau(a_0\delta_1(a_1)\delta_2(a_2 a_3)) - \tau(a_2 a_0 \delta_1(a_1)\delta_2(a_2)) \\
&= \tau(a_0 a_1 \delta_1(a_2)\delta_2(a_3)) - \tau(a_0\delta_1(a_1)a_2\delta_2(a_3)) - \tau(a_0 a_1\delta_1(a_2)\delta_2(a_3)) \\
&\quad + \tau(a_0\delta_1(a_1)\delta_2(a_2)a_3) + \tau(a_0\delta_1(a_1)a_2\delta_2(a_3)) - \tau(a_0\delta_1(a_1)\delta_2(a_2)a_2) = 0.
\end{aligned}$$

$\square$

Remark 2.16. The 0-cochain fitting in the pattern of $\phi_1, \phi_2, \phi_{12}$ is τ itself. In degree 0, cyclicity is trivial and a cocycle is the same as a trace (in the purely algebraic sense), so τ is also a cyclic cocycle.

These cyclic cocycles can also be obtained as the characters of rather simple cycles over $\tilde{\mathcal{A}}_\theta$ [6]. To this end, consider $\Omega_\theta := \tilde{\mathcal{A}}_\theta \otimes \Lambda^*\mathbb{C}^2$ with the graded algebra structure induced by $\Lambda^*\mathbb{C}^2$ (i. e. viewing $\tilde{\mathcal{A}}_\theta$ as a trivially graded algebra), but with a new graded derivation d defined by

$$d(a \otimes v) := \delta_1(a) \otimes (e_1 \wedge v) + \delta_2(a) \otimes (e_2 \wedge v), \quad a \in \tilde{\mathcal{A}}_\theta, v \in \Lambda^*\mathbb{C}^2. \tag{2.18}$$

Graded commutativity of the wedge product and the derivation property for δ_1, δ_2 ensure that d is a graded derivation, and $d^2 = 0$ is a consequence of Lemma 2.14(i). We now introduce four closed graded traces on Ω: any element $\omega \in \Omega$ can be written as

$$\omega = a_0 \otimes 1 + a_1 \otimes e_1 + a_2 \otimes e_2 + a_{12} \otimes (e_1 \wedge e_2)$$

for $a_0, a_1, a_2, a_{12} \in \tilde{\mathcal{A}}_\theta$. Letting j stand for 0, 1, 2, or 12, define

$$\int_j \omega := \tau(a_j),$$

which is easily seen to be closed and graded tracial. Moreover, comparison with Definition 2.15 immediately yields the following:

Lemma 2.17. *The cyclic cocycles $\tau, \phi_1, \phi_2, \phi_{12} \in HC^\bullet(\tilde{\mathcal{A}}_\theta)$ are the characters of the closed graded traces $\int_0, \int_1, \int_2, \int_{12}$ constructed above (in this order).*[9]

Remark 2.18 (Commutative case). In the commutative case $\theta = 0$, we have $\Omega_\theta \cong C^\infty(\mathbb{T}^2) \otimes \Lambda^2\mathbb{C}^2 \cong \Omega^*(\mathbb{T}^2)$, and this identifies d as defined in (2.18) with the exterior derivative (compare with Remark 2.13). Moreover, τ maps $f \in C^\infty(\mathbb{T}^2)$, expressed as a Fourier series $f(e^{2\pi i s}, e^{2\pi i t}) = \sum_{k,\ell} a_{k,\ell} e^{2\pi i(ks+\ell t)}$, to its constant part $a_{0,0}$. Hence, if $\mathrm{vol} \in \Omega^2(\mathbb{T}^2)$ denotes the standard volume form on $\mathbb{T}^2$ normalised to $\int_{\mathbb{T}^2} \mathrm{vol} = 1$, we can interpret τ as $\tau(f) = \int_{\mathbb{T}^2} f \,\mathrm{vol}$. This volume form corresponds to $(4\pi^2)^{-1} \otimes (e_1 \wedge e_2) \in \Omega_\theta$, and we thus obtain similar descriptions of the remaining cocycles. In particular, $\int_{12}$ corresponds to

$$\Omega^2(\mathbb{T}^2) \ni \omega \mapsto \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \omega,$$

so ϕ_{12} is (up to a factor) the standard cyclic cocycle obtained from integration as in (1.3).

2.5 Cyclic cohomology of $\tilde{\mathcal{A}}_\theta$

This section is devoted to the cyclic cohomology of non-commutative tori, or more precisely of the Fréchet algebras $\tilde{\mathcal{A}}_\theta$. In [13], cyclic cohomology is computed for non-commutative tori of arbitrary dimension, which are defined analogously to the two-dimensional case treated in this thesis. Here we follow the computation made in [13] while specifying to two-dimensional non-commutative tori, which simplifies several steps.

[9] To precisely follow definition 1.14, we are using the cycles $(\Omega, \rho, \int_j)$ where Ω is as above and ρ identifies $\tilde{\mathcal{A}}_\theta$ with $\tilde{\mathcal{A}}_\theta \otimes \Lambda^0\mathbb{C}^2 \subset \Omega$.

The strategy is to first describe the Hochschild cohomology $HH^\bullet(\tilde{\mathcal{A}}_\theta)$ via a suitable projective resolution, and then compute $HP(\tilde{\mathcal{A}}_\theta)$ from a spectral sequence as in Theorem 1.36. From there, one can extract $HC^\bullet(\tilde{\mathcal{A}}_\theta)$ without much difficulty. The results are collected in Theorem 2.22 below.

2.5.1 A projective resolution of $\tilde{\mathcal{A}}_\theta$

Here we describe a projective resolution of finite length for the Fréchet algebra $\tilde{\mathcal{A}}_\theta$, similar to a Koszul resolution of a commutative ring, following the presentation in [13]. As in the standard resolution (see subsection 1.3.1), this will be a complex of modules over the enveloping algebra $\tilde{\mathcal{A}}_\theta^e$.

Consider the sequence

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 = \tilde{\mathcal{A}}_\theta^e \xrightarrow{\epsilon} \tilde{\mathcal{A}}_\theta \longrightarrow 0, \tag{2.19}$$

where

$$P_m := \tilde{\mathcal{A}}_\theta^e \otimes \Lambda^m \mathbb{C}^2, \quad m = 0, 1, 2,$$

ϵ denotes multiplication $\tilde{\mathcal{A}}_\theta \otimes \tilde{\mathcal{A}}_\theta^{op} \to \tilde{\mathcal{A}}_\theta$, and the module morphisms d are defined (on bases) by

$$\begin{aligned} &d_2(1_e \otimes e_1 \wedge e_2) := (1_e - U^{-1} \otimes U) \otimes e_2 - (1_e - V^{-1} \otimes V) \otimes e_1, \\ &d_1(1_e \otimes e_1) := 1_e - U^{-1} \otimes U, \quad d_1(1_e \otimes e_2) := 1_e - V^{-1} \otimes V. \end{aligned}$$

Here $e_1 = (1,0)$, $e_2 = (0,1)$ are standard basis vectors of $\mathbb{C}^2$ and $1_e := 1 \otimes 1$ denotes the unit of the enveloping algebra.

The signs in the definition of d are chosen so that $d_1 \circ d_2 = 0$, and $\epsilon \circ d_1 = 0$ is also immediate from the construction. Thus (2.19) defines a chain complex of (free) $\tilde{\mathcal{A}}_\theta^e$-modules. Moreover, this complex is in fact acyclic. We show this here by exhibiting a contracting homotopy σ, but it also follows from the embedding of $P_\bullet$ into the standard complex, which will be discussed in subsection 2.5.3.[10] That is, σ consists of $\mathbb{C}$-linear maps $\sigma_{-1}\colon \tilde{\mathcal{A}}_\theta \to \tilde{\mathcal{A}}_\theta^e$, $\sigma_0\colon \tilde{\mathcal{A}}_\theta^e \to P_1$, $\sigma_1\colon P_1 \to P_2$. In degree -1, we simply have $\sigma_{-1}(a) := 1 \otimes a$ for $a \in \tilde{\mathcal{A}}_\theta$, whereas the other two are defined on tensor products of the monomials $U^r V^s$ (and extended uniquely). In their definition, we use the notation

$$\sum_{k=0}^{n}{}' := \begin{cases} \sum_{k=0}^{n}, & n \geq 0, \\ 0, & n = -1, \\ -\sum_{k=n+1}^{-1}, & n < -1, \end{cases} \tag{2.20}$$

adapted from [13]. Note also that all products denote the usual product of $\tilde{\mathcal{A}}_\theta$, even in the tensor factor corresponding to the opposite algebra.[11] Now, for $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ we

[10] The latter is how acyclicity is shown in [13], and the contracting homotopy given here was obtained by following the argument there.

[11] So we would write the product of $a_1 \otimes b_1$ and $a_2 \otimes b_2$ in the enveloping algebra as $a_1 a_2 \otimes b_2 b_1$.

set

$$\sigma_0(U^{r_1}V^{s_1}\otimes U^{r_2}V^{s_2}) := \sum_{k=0}^{r_1-1}{}'(U^{r_1-k}\otimes U^kV^{s_1}U^{r_2}V^{s_2})\otimes e_1 + \sum_{\ell=0}^{s_1-1}{}'(U^{r_1}V^{s_1-\ell}\otimes V^\ell U^{r_2}V^{s_2})\otimes e_2,$$

and

$$\sigma_1((U^{r_1}V^{s_1}\otimes U^{r_2}V^{s_2})\otimes e_1) := -\sum_{\ell=0}^{s_1-1}{}'(U^{r_1}V^{s_1-\ell}\otimes V^\ell U^{r_2}V^{s_2})\otimes e_1\wedge e_2,$$
$$\sigma_1((U^{r_1}V^{s_1}\otimes U^{r_2}V^{s_2})\otimes e_2) := 0.$$

We verify that this really defines a chain homotopy from $\mathrm{id}_{P_\bullet}$ to 0. Firstly,

$$\epsilon\sigma_{-1}(a) = 1a = a \quad \forall a\in\tilde{\mathcal{A}}_\theta.$$

Secondly, for all $r_1, s_2, r_2, s_2\in\mathbb{Z}$ we have

$$\begin{aligned}
&(d_1\sigma_0+\sigma_{-1}\epsilon)(U^{r_1}V^{s_1}\otimes U^{r_2}V^{s_2})\\
&=\sum_{k=0}^{r_1-1}{}'(U^{r_1-k}\otimes U^kV^{s_1}U^{r_2}V^{s_2})(1_e-U^{-1}\otimes U)\\
&\quad+\sum_{\ell=0}^{s_1-1}{}'(U^{r_1}V^{s_1-\ell}\otimes V^\ell U^{r_2}V^{s_2})(1_e-V^{-1}\otimes V)+1\otimes U^{r_1}V^{s_1}U^{r_2}V^{s_2}\\
&=U^{r_1}\otimes V^{s_1}U^{r_2}V^{s_2}-1\otimes U^{r_1}V^{s_1}U^{r_2}V^{s_2}+U^{r_1}V^{s_1}\otimes V^{s_2}U^{r_2}\\
&\quad-U^{r_1}\otimes V^{s_1}U^{r_2}V^{s_2}+1\otimes U^{r_1}V^{s_1}U^{r_2}V^{s_2}=U^{r_1}V^{s_1}\otimes V^{s_2}U^{r_2},
\end{aligned}$$

the definition of $\sum'$ ensuring that the telescoping sums work out correctly for all integer exponents.

Similar computations for $d_2\sigma_1+\sigma_0 d_1$, applied to basis elements, involve eight and four sums respectively. After applying the commutation relation (2.3) for some of them, they cancel in pairs almost completely. The result is $d_2\sigma_1+\sigma_0d_1=\mathrm{id}_{P_1}$.

Finally, one more telescoping sum computation yields $\sigma_1 d_2=\mathrm{id}_{P_2}$. Altogether, this shows that σ is a contracting homotopy, so $(P_\bullet, d)$ is acyclic and hence a projective resolution.

2.5.2 Hochschild cohomology of $\tilde{\mathcal{A}}_\theta$

Using the projective resolution (2.19), we can compute the Hochschild cohomology of $\tilde{\mathcal{A}}_\theta$ by applying the functor $\mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\cdot,\tilde{\mathcal{A}}_\theta^*)$. That is, we have to compute the cohomology of the cochain complex $\mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(P_\bullet,\tilde{\mathcal{A}}_\theta^*)$ whose coboundary maps are given by precomposition with $d_\bullet$.

We start with a description of the dual space $\tilde{\mathcal{A}}_\theta^*$. As was shown in Proposition 2.6, the map

$$\phi\colon \mathcal{S}(\mathbb{Z}^2) \to \tilde{\mathcal{A}}_\theta, \quad a \mapsto \sum_{k,\ell} a_{k,\ell} U^k V^\ell$$

is an isomorphism. Dualising ϕ then yields an isomorphism from $\tilde{\mathcal{A}}_\theta^*$ to the dual of $\mathcal{S}(\mathbb{Z}^2)$, whose elements are known as *tempered sequences*. Although an element of $\mathcal{S}(\mathbb{Z}^2)^*$ is formally a linear functional $f\colon \mathcal{S}(\mathbb{Z}^2) \to \mathbb{C}$, it can indeed be identified with the sequence $(f_{k,\ell})_{k,\ell\in\mathbb{Z}}$ given by $f_{k,\ell} := f(e_{k,\ell})$, where $e_{k,\ell} \in \mathcal{S}(\mathbb{Z}^2)$ is defined by $e_{k,\ell}(r,s) := \delta_{kr}\delta_{\ell s}$. Continuity of f is equivalent to $|f|$ being bounded by some positive linear combination of the seminorms defining the topology on $\mathcal{S}(\mathbb{Z}^2)$. Since these seminorms come in strictly increasing order of dominance, one of them suffices, so f is continuous if and only if there exist $N \in \mathbb{N}$ and $C \geq 0$ such that

$$|f(a)| \leq C\|a\|_N \quad \forall a \in \mathcal{S}(\mathbb{Z}^2).$$

As it is sufficient to have this estimate for a ranging over the $e_{k,\ell}$, we can say that the sequence $(f_{k,\ell})$ is tempered[12] if and only if there exist N, C as above such that

$$|f_{k,\ell}| \leq C(|k| + |\ell|)^N \quad \forall k, \ell \in \mathbb{Z}.$$

One can now similarly define tempered sequences with values in an arbitrary normed vector space X. We let $\mathrm{TS}(X)$ denote the space of all X*-valued tempered sequences*, that is, sequences $(x_{k,\ell})_{k,\ell\in\mathbb{Z}}$ in X such that there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$\|x_{k,\ell}\| \leq C(|k| + |\ell|)^N \quad \forall k, \ell \in \mathbb{Z}.$$

In other words, x is tempered if $\|\cdot\| \circ x$ is tempered in the usual sense.

Such tempered sequence spaces are in fact modules over $\tilde{\mathcal{A}}_\theta^e$. For $\tilde{\mathcal{A}}_\theta^*$ the action is given by

$$(a \otimes a') \cdot f(b) = f(a'ba) \quad \forall a, a', b \in \tilde{\mathcal{A}}_\theta, f \in \tilde{\mathcal{A}}_\theta^*.$$

This can be rephrased as

$$\begin{aligned}((U^{r_1}V^{s_1} \otimes U^{r_2}V^{s_2}) \cdot f)(U^k V^\ell) &= f(U^{r_2}V^{s_2}U^k V^\ell U^{r_1}V^{s_1}) \\ &= q^{\ell r_1 + r_1 s_2 + s_2 k} f(U^{k+r_1+r_2}V^{\ell+s_1+s_2}) \\ &\quad \forall r_1, r_2, s_1, s_2, k, \ell \in \mathbb{Z},\end{aligned}$$

so in the picture of sequences the action is

$$((U^{r_1}V^{s_1} \otimes U^{r_2}V^{s_2}) \cdot x)_{k,\ell} = q^{\ell r_1 + r_1 s_2 + s_2 k} x_{k+r_1+r_2, \ell+s_1+s_2}. \tag{2.21}$$

Observe that the action defined in (2.21) makes sense on all spaces of tempered sequences, so they all are modules over $\tilde{\mathcal{A}}_\theta^e$.

[12] i. e. describes a continuous linear functional on $\mathcal{S}(\mathbb{Z}^2)$

Here we shall use tempered sequences valued in $\Lambda^m\mathbb{C}^2$, with $m = 0, 1, 2$. An $\tilde{\mathcal{A}}_\theta^e$-linear map $f \in \mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\tilde{\mathcal{A}}_\theta^e \otimes \Lambda^m\mathbb{C}^2, \tilde{\mathcal{A}}_\theta^*)$ is uniquely determined by the elements $f(1_e \otimes e_I)$ where e_I denotes standard basis elements of $\Lambda^m\mathbb{C}^2$ (induced by the standard basis on $\mathbb{C}^2$). Each $f(1_e \otimes e_I) \in \tilde{\mathcal{A}}_\theta^*$ corresponds to a tempered sequence, and the same data can also be packaged into just one tempered sequence valued in $\Lambda^m\mathbb{C}^2$. Explicitly, we assign to f the tempered sequence $\psi_m(f) \in \mathrm{TS}(\Lambda^m\mathbb{C}^2)$ given by

$$(\psi_m(f))_{k,\ell} := \sum_I (f(1_e \otimes e_I))(U^k V^\ell) e_I \quad \forall k, \ell \in \mathbb{Z}. \tag{2.22}$$

This defines a map $\psi_m\colon \mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(P_m, \tilde{\mathcal{A}}_\theta^*) \to \mathrm{TS}(\Lambda^m\mathbb{C}^2)$ which is easily seen to be linear over $\tilde{\mathcal{A}}_\theta^e$ and bijective. Thus, we may replace $\mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(P_m, \tilde{\mathcal{A}}_\theta^*)$ with $\mathrm{TS}(\Lambda^m\mathbb{C}^2)$.

Having described the spaces in the dualised complex by tempered sequences, we now also translate the coboundary maps to this setting. For $m = 0$, take $x \in \mathrm{TS}(\mathbb{C})$. Then, for all $k, \ell \in \mathbb{Z}$,

$$\begin{aligned}
&\psi_1(\psi_0^{-1}(x) \circ d_1)_{k,\ell} \\
&= (\psi_0^{-1}(x) \circ d_1)(1_e \otimes e_1)(U^k V^\ell)e_1 + (\psi_0^{-1}(x) \circ d_1)(1_e \otimes e_2)(U^k V^\ell)e_2 \\
&= \psi_0^{-1}(x)(1_e - U^{-1} \otimes U)(U^k V^\ell)e_1 + \psi_0^{-1}(x)(1_e - V^{-1} \otimes V)(U^k V^\ell)e_2 \\
&= ((1_e - U^{-1} \otimes U) \cdot x)_{k,\ell} e_1 + ((1_e - V^{-1} \otimes V) \cdot x)_{k,\ell} e_2 \\
&= (1 - q^{-\ell}) x_{k,\ell}\, e_1 + (1 - q^k) x_{k,\ell}\, e_2.
\end{aligned}$$

A similar computation shows that for $x \in \mathrm{TS}(\mathbb{C}^2)$, written as $x^{(1)}e_1 + x^{(2)}e_2$ for $x^{(1)}, x^{(2)} \in \mathrm{TS}(\mathbb{C})$, we have

$$\psi_2(\psi_1^{-1} \circ d_2)_{k,\ell} = \big((1 - q^{-\ell}) x_{k,\ell}^{(2)} - (1 - q^k) x_{k,\ell}^{(1)}\big) e_1 \wedge e_2.$$

Let us note that both maps can be interpreted in terms of the wedge product in the exterior algebra $\Lambda^*\mathbb{C}^2$, which lifts to sequences in $\Lambda^*\mathbb{C}^2$ by applying it entry-wise. If we denote by $\xi \in TS(\Lambda^*\mathbb{C}^2)$ the sequence

$$\xi_{k,\ell} := (1 - q^{-\ell})e_1 + (1 - q^k)e_2 \quad \forall k, \ell \in \mathbb{Z}, \tag{2.23}$$

then the coboundary operator in the tempered sequence setting is simply

$$\xi \wedge (\cdot)\colon\ TS(\Lambda^*\mathbb{C}^2) \to TS(\Lambda^*\mathbb{C}^2)$$

(restricted and corestricted appropriately). Note that although ξ is not tempered, it is bounded, so $\xi \wedge x$ is tempered whenever x is.

We now compute the Hochschild cohomology of $\tilde{\mathcal{A}}_\theta$ as the cohomology of the complex

$$0 \longleftarrow \mathrm{TS}(\Lambda^2\mathbb{C}^2) \xleftarrow{\xi\wedge\cdot} \mathrm{TS}(\Lambda^1\mathbb{C}^2) \xleftarrow{\xi\wedge\cdot} \mathrm{TS}(\Lambda^0\mathbb{C}^2) \longleftarrow 0$$

Like many aspects of rotation algebras, the result depends on whether θ is rational. In case θ is rational, we always write it as m/n with $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $\gcd(m, n) = 1$.

Since the wedge product in $\mathrm{TS}(\Lambda^*\mathbb{C}^2)$ is an entry-wise operation, both the kernel and the image of $\xi \wedge \cdot$ are described by separate equations for every index pair (k,ℓ). The equations at k,ℓ become trivial precisely when $\xi_{k,\ell} = 0$, i. e. when $\ell\theta$ and $k\theta$ are integers. If θ is irrational, this happens only for $k = \ell = 0$, whereas for $\theta = m/n$ all indices nr, ns with $r, s \in \mathbb{Z}$ fulfil this condition. Triviality of the describing equations means that any tempered sequence supported only at indices where ξ vanishes is a cocycle.

Let us now turn to indices (k,ℓ) such that $\xi_{k,\ell} \neq 0$. Then the equation

$$\xi_{k,\ell} \wedge \underline{x} = 0$$

in $\Lambda^*\mathbb{C}$ is solved by precisely those $\underline{x} \in \Lambda^*\mathbb{C}^2$ of the form $\underline{x} = \xi_{k,\ell} \wedge \underline{y}$ for some $\underline{y} \in \Lambda^*\mathbb{C}^2$. It follows that a cocycle $x \in \mathrm{TS}(\Lambda^*\mathbb{C}^2)$ must be of the form $x = x'\xi \wedge y$ for a not necessarily tempered sequence $y = (y_{k,\ell})_{k,\ell\in\mathbb{Z}}$ with values in $\Lambda^*\mathbb{C}^2$, and $x' \in \mathrm{TS}(\Lambda^*\mathbb{C})$ supported only at the zeros of ξ. Such a cocycle vanishes in cohomology precisely when $x' = 0$ and y can be chosen to be a tempered sequence. Hence we have:

Theorem 2.19 (cf. [13], Theorem 4.1)**.** *The Hochschild cohomology $HH^\bullet(\tilde{\mathcal{A}}_\theta)$ is isomorphic to the direct sum of*

1. *the subspace of $\mathrm{TS}(\Lambda^*\mathbb{C}^2)$ consisting of all $x \in \mathrm{TS}(\Lambda^*\mathbb{C}^2)$ for which $x_{k,\ell} = 0$ except when $\theta k, \theta\ell \in \mathbb{Z}$, and*

2. *the space of all tempered sequences in $\mathrm{TS}(\Lambda^*\mathbb{C}^2)$ of the form $x = \xi \wedge y$, where $y = (y_{k,\ell})_{k,\ell\in\mathbb{Z}}$ has values in $\Lambda^*(\mathbb{C}^2)$, quotiented by the subspace of all such x for which y may be chosen to be tempered,*

with the grading induced by the usual grading of $\Lambda^*\mathbb{C}^2$.

Remark 2.20.

1. In particular, $HH^n(\tilde{\mathcal{A}}_\theta) \cong 0$ for $n > 2$.

2. The first direct summand in Theorem 2.19 is isomorphic to $\mathbb{C}$ if θ is irrational, and to $\mathrm{TS}(\Lambda^*\mathbb{C}^2)$ otherwise, the latter via sending x to y defined by $y_{r,s} \coloneqq x_{nr,ns}$.

3. If we normalise y in the second item of Theorem 2.19 by imposing $y_{k,\ell} = 0$ whenever $k\theta, \ell\theta \in \mathbb{Z}$, then it will be uniquely determined (by the sequence $\xi \wedge y$).

4. For $\theta = 0$, the Hochschild cohomology of $\tilde{\mathcal{A}}_0 \cong C^\infty(\mathbb{T}^2)$ is isomorphic to the space of de Rham currents of $\mathbb{T}^2$ (cf. [5]). In terms of our computation, $\theta = 0$ implies $\xi = 0$, so the Hochschild cohomology is fully captured by the first direct summand in Theorem 2.19. By item 2 of this remark, we then have $HH^k(C^\infty(\mathbb{T}^2)) \cong \mathrm{TS}(\Lambda^k\mathbb{C}^2)$. Translating back from the $\mathcal{S}(\mathbb{Z}^2)$ picture to smooth functions, this can be seen to coincide with the dual space of $\Omega^k(\mathbb{T}^2)$ as expected.

2.5.3 Computation of S, I, B spectral sequence

Equipped with a description of the Hochschild cohomology, we now aim to compute the periodic cyclic cohomology of $\tilde{\mathcal{A}}_\theta$, using the tools from subsection 1.3.4, that is, the spectral sequence associated to the exact couple

$$\begin{tikzcd} HC^\bullet(\tilde{\mathcal{A}}_\theta) \arrow[rr, "S"] & & HC^\bullet(\tilde{\mathcal{A}}_\theta) \arrow[dl, "I"] \\ & HH^\bullet(\tilde{\mathcal{A}}_\theta) \arrow[ul, "B"] & \end{tikzcd}$$

Before going into any computations, let us observe that this spectral sequence collapses on the second page. To see this, note that any nonzero element of $\operatorname{im} S$ has degree at least 2. Now for $n > 2$ we have $B^n = 0$ (because its domain is zero), so $\operatorname{im} B^n = 0$ and thus $\ker S^{n-1} = 0$ by Theorem 1.30. In other words, S is injective on $\bigoplus_{n\geq 2} HC^n_\lambda(\tilde{\mathcal{A}}_\theta)$ and thus on $\operatorname{im} S$. But this means that the induced map S' in the derived couple is injective, so $\operatorname{im} B' = \ker S' = 0$, which yields $B' = 0$ and thus $d_2 = I'B' = 0$. A similar argument works for every subsequent page.

Hence, it is sufficient to compute the second page, that is, the cohomology of $HH^\bullet(\tilde{\mathcal{A}}_\theta)$ with respect to IB. Note that IB can be defined on the cochain level, where it corresponds to viewing B as a map $C^\bullet(\tilde{\mathcal{A}}_\theta) \to C^\bullet(\tilde{\mathcal{A}}_\theta)$. To simplify notation, we will denote this operator IB on cochains simply by B.

We now have to translate the operator B acting on Hochschild cochains to the picture of tempered sequences, in which we have described the Hochschild cohomology. In [13], an embedding of $P_\bullet$ into the standard complex (see subsection 1.3.1) is given explicitly. More precisely, this is an injective map of complexes $h_\bullet$; it consists of $h_{-1} = \mathrm{id}$, $h_0 = \mathrm{id}$, and the $\tilde{\mathcal{A}}^e_\theta$-linear maps

$$h_m \colon \tilde{\mathcal{A}}^e_\theta \otimes \Lambda^m \mathbb{C}^2 \to \tilde{\mathcal{A}}^e_\theta \otimes (\tilde{\mathcal{A}}_\theta)^{\otimes m}, \quad m = 1, 2,$$

defined by

$$\begin{aligned} h_2(1_e \otimes e_1 \wedge e_2) &\coloneqq (U^{-1}V^{-1} \otimes 1) \otimes V \otimes U - (V^{-1}U^{-1} \otimes 1) \otimes U \otimes V, \\ h_1(1_e \otimes e_1) &\coloneqq (U^{-1} \otimes 1) \otimes U, \quad h_1(1_e \otimes e_2) \coloneqq (V^{-1} \otimes 1) \otimes V. \end{aligned}$$

A (continuous) left inverse of this embedding is provided by $\kappa_\bullet$, defined by

$$\kappa_1(1_e \otimes U^r V^s) \coloneqq \sum_{k=0}^{r-1}{}'(U^{r-k} \otimes U^k V^s) \otimes e_1 + \sum_{\ell=0}^{s-1}{}'(U^r V^{s-\ell} \otimes V^\ell) \otimes e_2$$

for all $r, s \in \mathbb{Z}$ and

$$\kappa_2(1_e \otimes U^{r_1}V^{s_1} \otimes U^{r_2}V^{s_2}) \coloneqq \sum_{\ell=0}^{s_1-1}{}' \sum_{k=0}^{r_2-1}{}'(U^{r_1}V^{s_1}U^{r_2-k}V^{-\ell} \otimes V^\ell U^k V^{s_2}) \otimes e_1 \wedge e_2$$

for all $r_1, s_1, r_2, s_2 \in \mathbb{Z}$, using the primed sum notation (2.20).

Lastly, one has to be aware of two different versions of the Hochschild complex appearing here. One of them is $C^n(\tilde{\mathcal{A}}_\theta, b)$ as per Definition 1.13. The other version is the immediate result of applying the functor $\operatorname{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\cdot, \tilde{\mathcal{A}}_\theta^*)$ to the standard resolution of $\tilde{\mathcal{A}}_\theta$. We shall denote it as

$$\tilde{C}^n(\tilde{\mathcal{A}}_\theta) \coloneqq \operatorname{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\tilde{\mathcal{A}}_\theta^e \otimes \tilde{\mathcal{A}}_\theta^{\otimes n}, \tilde{\mathcal{A}}_\theta^*), \quad n \in \mathbb{N},$$

with coboundary maps denoted $\tilde{b}^\bullet$ obtained from the boundary maps of the standard resolution. They are identified by the isomorphism of cochain complexes (1.4).

The overall situation on the cochain level is summarised in the following diagram, where we have denoted the action of the functor $\operatorname{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\cdot, \tilde{\mathcal{A}}_\theta^*)$ with an asterisk:

$$\begin{tikzcd}
\cdots & C^2(\tilde{\mathcal{A}}_\theta) \arrow[l] \arrow[d, "\cong"] & C^1(\tilde{\mathcal{A}}_\theta) \arrow[l, "b"'] \arrow[d, "\cong"] & C^0(\tilde{\mathcal{A}}_\theta) \arrow[l, "b"'] \arrow[d, "\cong"] & 0 \arrow[l] \\
\cdots & \tilde{C}^2(\tilde{\mathcal{A}}_\theta) \arrow[l] \arrow[d, shift left, "h_2^*"] & \tilde{C}^1(\tilde{\mathcal{A}}_\theta) \arrow[l, "\tilde{b}"'] \arrow[d, shift left, "h_1^*"] & \tilde{C}^0(\tilde{\mathcal{A}}_\theta) \arrow[l, "\tilde{b}"'] \arrow[d, equal] & 0 \arrow[l] \\
0 & P_2^* \arrow[l] \arrow[u, shift left, "\kappa_2^*"] \arrow[d, "\psi_2"', "\cong"] & P_1^* \arrow[l, "d_2^*"'] \arrow[u, shift left, "\kappa_1^*"] \arrow[d, "\psi_1"', "\cong"] & P_0^* \arrow[l, "d_1^*"'] \arrow[d, "\psi_0"', "\cong"] & 0 \arrow[l] \\
0 & \mathrm{TS}(\Lambda^2\mathbb{C}^2) \arrow[l] & \mathrm{TS}(\Lambda^1\mathbb{C}^2) \arrow[l, "\xi\wedge\cdot"'] & \mathrm{TS}(\Lambda^0\mathbb{C}^2) \arrow[l, "\xi\wedge\cdot"'] & 0 \arrow[l]
\end{tikzcd} \tag{2.24}$$

Conjugation with the isomorphism $C^\bullet(\tilde{\mathcal{A}}_\theta) \xrightarrow{\cong} \tilde{C}^\bullet(\tilde{\mathcal{A}}_\theta)$ transforms B (as defined in (1.9)) to the operator $\tilde{B}$, given explicitly by

$$\tilde{B}^1 f(1_e)(a_0) = f(1_e \otimes a_0)(1) + f(1_e \otimes 1)(a_0), \tag{2.25}$$

$$\begin{aligned}\tilde{B}^2 g(1_e \otimes a_1)(a_0) &= g(1_e \otimes a_0 \otimes a_1)(1) - g(1_e \otimes a_1 \otimes a_0)(1) \\ &\quad - g(1_e \otimes a_1 \otimes 1)(a_0) + g(1_e \otimes a_0 \otimes 1)(a_1)\end{aligned} \tag{2.26}$$

for $f \in \tilde{C}^1(\tilde{\mathcal{A}}_\theta)$ $g \in \tilde{C}^2(\tilde{\mathcal{A}}_\theta)$ and $a_0, a_1 \in \tilde{\mathcal{A}}_\theta$.

We now describe $\tilde{B}$ in the setting of tempered sequences. Firstly, let $x = x^{(1)}e_1 + x^{(2)}e_2 \in \mathrm{TS}(\Lambda^1\mathbb{C}^2)$. Then $f \coloneqq \kappa_1^*\psi_1^{-1}(x) \in \tilde{C}^1(\tilde{\mathcal{A}}_\theta)$ is given by

$$\begin{aligned}f(1_e \otimes U^{r_1}V^{s_1})(U^{r_2}V^{s_2}) &= \sum_{k=0}^{r_1-1}{}' q^{r_2 s_1 + (r_1-k)(s_1+s_2)} x^{(1)}_{r_1+r_2, s_1+s_2} \\ &\quad + \sum_{\ell=0}^{s_1-1}{}' q^{(r_1+r_2)\ell + r_1 s_2} x^{(2)}_{r_1+r_2, s_1+s_2}, \quad r_1, r_2, s_1, s_2 \in \mathbb{Z},\end{aligned}$$

so the map corresponding to $\tilde{B}^1$ sends x to

$$\begin{aligned}(\psi_0 \tilde{B}^1(f))_{r,s} &= \tilde{B}^1(f)(1_e)(U^r V^s)\\ &= f(1_e \otimes U^r V^s)(U^0 V^0) + f(1_e \otimes U^0 V^0)(U^r V^s)\\ &= \sum_{k=0}^{r-1}{}' q^{(r-k)s} x^{(1)}_{r,s} + \sum_{\ell=0}^{s}{}' q^{r\ell} x^{(2)}_{r,s}, \quad r,s \in \mathbb{Z},\end{aligned} \tag{2.27}$$

since the second term in the second line vanishes.

For $\tilde{B}^2$, take $x \in \mathrm{TS}(\mathbb{C})$ and view it as the sequence $xe_1 \wedge e_2 \in \mathrm{TS}(\Lambda^2\mathbb{C}^2)$ (also denoted x). It corresponds to $g = \kappa_2^* \psi_2^{-1}(x) \in \tilde{C}^2(\tilde{\mathcal{A}}_\theta)$, with

$$\begin{aligned}&g(1_e \otimes U^{r_1} V^{s_1} \otimes U^{r_2} V^{s_2})(U^{r_3} V^{s_3})\\ &\quad = \sum_{\ell=0}^{s_1-1}{}' \sum_{k=0}^{r_2-1}{}' q^{k\ell + r_3(\ell+s_2) + r_1(\ell+s_2+s_3) + (r_2-k)(\ell+s_1+s_2+s_3)} x_{r_1+r_2+r_3, s_1+s_2+s_3},\\ &\quad r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z}.\end{aligned} \tag{2.28}$$

In contrast to h_0, h_1 is not the identity; we compute

$$\begin{aligned}&(\psi_1 h_1^* \tilde{B}^2(g))_{r,s}\\ &= h_1^* \tilde{B}^2(g)(1_e \otimes e_1)(U^r V^s) e_1 + h_1^* \tilde{B}^2(g)(1_e \otimes e_2)(U^r V^s) e_2\\ &= \tilde{B}^2(g)((U^{-1} \otimes 1) \otimes U)(U^r V^s) e_1 + \tilde{B}^2(g)((V^{-1} \otimes 1) \otimes V)(U^r V^s) e_2\\ &= q^{-s} \tilde{B}^2(g)(1_e \otimes U^1 V^0)(U^{r-1} V^s) e_1 + \tilde{B}^2(g)(1_e \otimes U^0 V^1)(U^r V^{s-1}) e_2.\end{aligned}$$

Applying formula (2.26), one finds that each time only one of its four summands is nonzero. The exponents of q in (2.28) also simplify drastically, and one is left with

$$(\psi_1 h_1^* \tilde{B}^2(g))_{r,s} = \sum_{\ell=0}^{s-1}{}' q^{r\ell} x_{r,s} e_1 - \sum_{k=0}^{r-1}{}' q^{(r-k)s} x_{r,s} e_2. \tag{2.29}$$

Comparing (2.27) and (2.29), one is led to consider the $\Lambda^*\mathbb{C}^2$-valued sequence $\eta = (\eta_{r,s})_{r,s\in\mathbb{Z}}$ defined by

$$\eta_{r,s} := \left(\sum_{k=0}^{r-1}{}' q^{(r-k)s} \right) e_1 + \left(\sum_{\ell=0}^{s-1}{}' q^{r\ell} \right) e_2. \tag{2.30}$$

To describe the results in terms of η, let us briefly recall some linear algebra. The vector space $\Lambda^*\mathbb{C}^2$ has a standard inner product, namely the one for which $\{1, e_1, e_2, e_1 \wedge e_2\}$ is an orthonormal basis. Given $v \in \Lambda^*\mathbb{C}^2$, we may consider the endomorphism $T_v \in \mathrm{End}(\Lambda^*\mathbb{C}^2)$ given by $x \mapsto v \wedge x$, and in particular form its adjoint T_v^*. Its matrix representation can be read off from the identity

$$\langle T_v^*(e_I), e_J \rangle = \langle e_I, v \wedge e_J \rangle \quad \forall I, J \subseteq \{1,2\}.$$

Now if $v = ae_1 + be_2$, then one finds that

$$T_v^*(1) = 0, \quad T_v^*(e_1) = a, \quad T_v^*(e_2) = b, \quad T_v^*(e_1 \wedge e_2) = -be_1 + ae_2,$$

and from this the relation to equations (2.27) and (2.29) is clearly recognisable. Passing back to sequences, we define $T_\eta^* \in \mathrm{End}(\mathrm{TS}(\Lambda^*\mathbb{C}^2))$ entry-wise as acting by $T_{\eta_{r,s}}$ on the (r,s)-entry. We can now express our result as

$$(\psi_{m-1} \circ h_{m-1}^* \circ \tilde{B}^m \circ \kappa_m^* \circ \psi_m^{-1})(x) = T_\eta^*(x), \quad m = 1, 2,\ x \in \mathrm{TS}(\Lambda^m\mathbb{C}^2).$$

With this we have completely translated B into the tempered sequence setting. On the cochain level the result is represented by the following diagram, which fully commutes in cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & C^2(\tilde{\mathcal{A}}_\theta) \arrow[r, "B"] \arrow[d, "\cong"] & C^1(\tilde{\mathcal{A}}_\theta) \arrow[r, "B"] \arrow[d, "\cong"] & C^0(\tilde{\mathcal{A}}_\theta) \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & \tilde{C}^2(\tilde{\mathcal{A}}_\theta) \arrow[r, "\tilde{B}"] \arrow[d, shift left, "h_2^*"] & \tilde{C}^1(\tilde{\mathcal{A}}_\theta) \arrow[r, "\tilde{B}"] \arrow[d, shift left, "h_1^*"] & \tilde{C}^0(\tilde{\mathcal{A}}_\theta) \arrow[r] \arrow[d, equal] & 0 \\
 & P_2^* \arrow[u, shift left, "\kappa_2^*"] \arrow[d, "\psi_2"', "\cong"] & P_1^* \arrow[u, shift left, "\kappa_1^*"] \arrow[d, "\psi_1"', "\cong"] & P_0^* \arrow[d, "\psi_0"', "\cong"] & \\
0 \arrow[r] & \mathrm{TS}(\Lambda^2\mathbb{C}^2) \arrow[r, "T_\eta^*"] & \mathrm{TS}(\Lambda^1\mathbb{C}^2) \arrow[r, "T_\eta^*"] & \mathrm{TS}(\Lambda^0\mathbb{C}^2) \arrow[r] & 0
\end{tikzcd}$$

To unify notation, we now also write $\xi \wedge \cdot$ as T_ξ. Computing the second page of our spectral sequence amounts to computing the cohomology of $H^\bullet(\mathrm{TS}(\Lambda^*\mathbb{C}^2), T_\xi)$ with respect to the coboundary T_η^*.

As before, this initially decomposes into entry-wise linear algebra problems for each index (r,s), which depend on whether $\xi_{r,s}$ and $\eta_{r,s}$ vanish. First of all, for $r = s = 0$ we always have $\xi_{0,0} = \eta_{0,0} = 0$. If $(r,s) \neq (0,0)$, then we claim that one of $\xi_{r,s}$ and $\eta_{r,s}$ is always nonzero. Indeed, suppose $\xi_{r,s} = 0$. Then we must have $\theta = m/n \in \mathbb{Q}$ (written in lowest terms), and $r, s \in n\mathbb{Z}$. But then the summands $q^{(r-k)s}$ and $q^{r\ell}$ appearing in the definition (2.30) of $\eta_{r,s}$ are all equal to 1. Since $(r,s) \neq (0,0)$, one of the sums is nonempty, so $\eta_{r,s} \neq 0$.

We will in fact need something stronger in this direction. The following is analogous to Lemma 5.1 in [13] and Lemma 52 in [5]:

Lemma 2.21.

$$\|\xi_{r,s}\| + \|\eta_{r,s}\| \geq \frac{\pi}{4} \frac{1}{|r| + |s|} \quad \forall (r,s) \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

Proof. Let $(r,s) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. From the definition (2.30) of η, one computes that $\|\eta_{r,s}\|^2 = A + B$, where

$$A = \begin{cases} |1-q^{rs}|^2/|1-q^s|^2, & q^s \neq 1, \\ r^2, & q^s = 1, \end{cases} \qquad B = \begin{cases} |1-q^{rs}|^2/|1-q^r|^2, & q^r \neq 1, \\ s^2, & q^r = 1, \end{cases}$$

and from (2.23) one directly reads off that

$$\|\xi_{r,s}\|^2 = |1-q^r|^2 + |1-q^s|^2.$$

Whenever $q^r = 1$ or $q^s = 1$ we have $A + B \geq 1$ and the claim follows, so assume from now on that $q^r, q^s \neq 1$. Then

$$\|\xi_{r,s}\| + \|\eta_{r,s}\| \geq \frac{1}{\sqrt{2}} \left(\frac{|1-q^{rs}|}{|1-q^s|} + \frac{|1-q^{rs}|}{|1-q^r|} + |1-q^r| + |1-q^s| \right).$$

Now express $q^r \in S^1$ as $q^r = e^{i\alpha}$ with $\alpha \in [-\pi, \pi)$. Then $A \geq 1$ if $\operatorname{dist}(s\alpha, \mathbb{Z}) \geq |\alpha|$, and this is certainly the case if $s\alpha \in [-\pi, \pi]$.[13] We may thus assume that $|s\alpha| \geq \pi$ or equivalently $|\alpha| \geq \pi/|s|$, from which it follows that

$$|1-q^r| = \sqrt{2 - 2\cos\alpha} \geq \sqrt{2 - 2\cos\pi/|s|} \geq \pi/(2|s|)$$

(using $1 - \cos x \leq x^2/8$ for $x \in [-\pi, \pi]$). Together with the analogous statement for s this yields the claimed inequality.[14] □

Lastly, it is useful to observe that ξ and η are perpendicular. This can be checked directly from the definitions, where it becomes a telescoping sum computation. However, it also follows from the fact that T_η^* has to map the T_ξ-coboundary ξ to another T_ξ-coboundary, hence to zero for degree reasons. Indeed, this implies that

$$0 = \langle T_\eta^*(\xi), 1 \rangle = \langle \xi, \eta \rangle. \tag{2.31}$$

Now let $(r,s) \in \mathbb{Z}^2 \setminus \{0,0\}$ and $\underline{x} \in \Lambda^0\mathbb{C}^2$. We can have $\underline{x} \in \ker T_{\xi_{r,s}}$ only when $\xi_{r,s} = 0$. Then $\eta_{r,s} \neq 0$, and one easily concludes that $\underline{x} = T_\eta^* \underline{y}$ for some $\underline{y} \in \Lambda^1\mathbb{C}^2$. More precisely, assuming $\eta_{r,s}^{(1)} \neq 0$, we may take $\underline{y} = x/\eta_{r,s}^{(1)} e_1$, which also establishes $\|\underline{y}\| \leq \|\underline{x}\|$ since $\xi_{r,s} = 0$ implies $\eta_{r,s}^{(1)} \in \mathbb{Z}$. (If $\eta_{r,s}^{(1)} = 0$, use a similar construction with the other component.) Having assumed $\xi_{r,s} = 0$, this $\underline{y}$ is (trivially) contained in $\ker T_{\xi_{r,s}}$. This argument lifts to tempered sequences without problems: if $x \in \mathrm{TS}(\Lambda^0\mathbb{C}^2) \cap \ker T_\xi$, then the above construction yields $y \in \mathrm{TS}(\Lambda^0\mathbb{C}^2) \cap \ker T_\xi$ such that $x - T_\eta^* y$ is supported only in the $(0,0)$ entry. Hence

$$H^0(H^\bullet(\mathrm{TS}(\Lambda^*\mathbb{C}^2), T_\xi), T_\eta^*) \cong \mathbb{C}. \tag{2.32}$$

[13] Visually, this is the case where $q^r, q^{2r}, \ldots, q^{rs}$ does not wrap around the unit circle.

[14] In fact with $2\sqrt{2}$ in place of 4, but we used $\pi/4 \leq 1$ earlier in the proof.

Let us now pass to $\Lambda^1\mathbb{C}^2$. To avoid excessive case distinctions, let $\{v_1, v_2\} \subseteq \Lambda^1\mathbb{C}^2$ be an orthonormal basis such that $\xi_{r,s} = \|\xi_{r,s}\| v_1$ and $\eta_{r,s} = \|\eta_{r,s}\| v_2$. We have

$$\ker T_{\xi_{r,s}} \cap \Lambda^1\mathbb{C}^2 = \begin{cases} \operatorname{span}\{v_1\}, & \xi_{r,s} \neq 0, \\ \Lambda^1\mathbb{C}^2, & \xi_{r,s} = 0, \end{cases}$$

and, using $\ker T^*_{\eta_{r,s}} = (\operatorname{im} T_{\eta_{r,s}})^\perp$ and (2.31),

$$\ker T^*_{\eta_{r,s}} \cap \Lambda^1\mathbb{C}^2 = \begin{cases} \operatorname{span}\{v_1\}, & \eta_{r,s} \neq 0, \\ \Lambda^1\mathbb{C}^2, & \eta_{r,s} = 0. \end{cases}$$

Similarly, one finds that

$$\operatorname{im} T_{\xi_{r,s}} \cap \Lambda^1\mathbb{C}^2 = \begin{cases} \operatorname{span}\{v_1\}, & \xi_{r,s} \neq 0, \\ \{0\}, & \xi_{r,s} = 0, \end{cases}$$

and

$$\operatorname{im} T^*_{\eta_{r,s}} \cap \Lambda^1\mathbb{C}^2 = \begin{cases} \operatorname{span}\{v_1\}, & \eta_{r,s} \neq 0, \\ \{0\}, & \eta_{r,s} = 0. \end{cases}$$

Hence, if $(r, s) \neq (0, 0)$ and $\underline{x} \in \Lambda^1\mathbb{C}^2 \cap \ker T_{\xi_{r,s}} \cap \ker T^*_{\eta_{r,s}}$, then $\underline{x} = a\|\underline{x}\| v_1$ for $a \in S^1$, and we can express it as

$$\underline{x} = T_{\xi_{r,s}}(\underline{y}) + T^*_{\eta_{r,s}}(\underline{z})$$

for $\underline{y} \in \Lambda^0\mathbb{C}^2$ and $\underline{z} \in \Lambda^2\mathbb{C}^2$.

In order to lift this to tempered sequences, let us take a closer look at the possible norms of $\underline{y}, \underline{z}$ as above. Identifying $\underline{z}$ with a complex number via $\Lambda^2\mathbb{C}^2 \xrightarrow{\cong} \mathbb{C}$, we have

$$a\|\underline{x}\| v_1 = T_{\xi_{r,s}}(\underline{y}) + T^*_{\eta_{r,s}}(\underline{z}) = (\underline{y}\|\xi_{r,s}\| + b\underline{z}\|\eta_{r,s}\|) v_1$$

(where $b \in S^1$ is such that $T^*_{\eta_{r,s}}(e_1 \wedge e_2) = b\|\eta_{r,s}\| v_1$), and the only condition on $\underline{y}, \underline{z}$ is that

$$\underline{y}\|\xi_{r,s}\| + b\underline{z}\|\eta_{r,s}\| \stackrel{!}{=} a\|\underline{x}\|.$$

Choosing appropriate phases, this further reduces to

$$|\underline{y}|\|\xi_{r,s}\| + |\underline{z}|\|\eta_{r,s}\| \stackrel{!}{=} \|\underline{x}\|,$$

and a straightforward attempt is to try $|\underline{y}| = |\underline{z}| =: \rho$, which is then determined by

$$\rho = (\|\xi_{r,s}\| + \|\eta_{r,s}\|)^{-1}\|\underline{x}\|/2. \tag{2.33}$$

Now, if $x \in \mathrm{TS}(\Lambda^1\mathbb{C}^2) \cap \ker T_\xi \cap \ker T^*_\eta$, then by the argument above there are sequences y, z such that $x - T_\xi y - T^*_\eta z$ is supported only at $(0, 0)$. Moreover, according to (2.33) and Lemma 2.21, they can be chosen to be tempered (because dividing the tempered sequence $(\|x_{r,s}\|)_{r,s}$ by $(|r|+|s|)_{r,s}$ again yields a tempered sequence). We conclude

$$H^1(H^\bullet(\mathrm{TS}(\Lambda^*\mathbb{C}^2), T_\xi), T^*_\eta) \cong \Lambda^1\mathbb{C}^2. \tag{2.34}$$

Finally, $T^*_{\eta_{r,s}}$ is injective on $\Lambda^2\mathbb{C}^2$ whenever $\eta_{r,s} \neq 0$. Thus, $x \in \mathrm{TS}(\Lambda^2\mathbb{C}^2) \cap \ker T^*_\eta$ can only have nonzero entries at (r,s) such that $\eta_{r,s} = 0$. Apart from $(r,s) = (0,0)$, we have $\xi_{r,s} \neq 0$ at all such (r,s). Thus, one can construct a sequence y valued in $\Lambda^1\mathbb{C}^2$ such that $x - T_\xi y$ is supported only at $(0,0)$, and, as before, Lemma 2.21 ensures that y may be chosen tempered. It follows that

$$H^2(H^*(\mathrm{TS}(\Lambda^\bullet\mathbb{C}^2), T_\xi), T^*_\eta) \cong \Lambda^2\mathbb{C}^2. \tag{2.35}$$

We can now fully describe the periodic cyclic cohomology of $\tilde{\mathcal{A}}_\theta$, and also deduce the cyclic cohomology from it:

Theorem 2.22 (cf. [13], Theorems 5.4 and 6.1)**.** *Let* $\theta \in [0,1)$.

(i) $HP(\tilde{\mathcal{A}}_\theta) \cong \Lambda^*\mathbb{C}^2$ *with corresponding gradings. Here the degree of a class in* $HP(\tilde{\mathcal{A}}_\theta)$ *is the lowest cocycle degree in which a representative exists. In particular,* $HC^{ev}(\tilde{\mathcal{A}}_\theta) \cong \mathbb{C} \oplus \mathbb{C}$ *and* $HC^{odd}(\tilde{\mathcal{A}}_\theta) \cong \mathbb{C}^2$ *as graded vector spaces.*

(ii) $H^{ev}(\tilde{\mathcal{A}}_\theta)$ *is generated by the equivalence classes of* $\tau \in Z^0_\lambda(\tilde{\mathcal{A}}_\theta)$ *and* $\phi_{12} \in Z^2_\lambda(\tilde{\mathcal{A}}_\theta)$*.* $H^{odd}(\tilde{\mathcal{A}}_\theta)$ *is generated by the equivalence classes of* $\phi_1, \phi_2 \in Z^1_\lambda(\tilde{\mathcal{A}}_\theta)$*. (See Definition 2.15.)*

(iii) The cyclic cohomology is given by

$$\begin{aligned}
HC^0(\tilde{\mathcal{A}}_\theta) &= \mathrm{span}\{[\tau]\} \oplus \ker S^0,\\
HC^1(\tilde{\mathcal{A}}_\theta) &= \mathrm{span}\{[\phi_1],[\phi_2]\} \oplus \ker S^1,\\
HC^2(\tilde{\mathcal{A}}_\theta) &= \mathrm{span}\{[\phi_{12}]\} \oplus \mathrm{im}\, S^0 = \mathrm{span}\{[\phi_{12}],[S\tau]\},\\
HC^n(\tilde{\mathcal{A}}_\theta) &\cong HC^{ev}(\tilde{\mathcal{A}}_\theta) \quad n \geq 2 \text{ even},\\
HC^n(\tilde{\mathcal{A}}_\theta) &\cong HC^{odd}(\tilde{\mathcal{A}}_\theta) \quad n \geq 3 \text{ odd}.
\end{aligned}$$

Proof. Part (i) follows from (2.32), (2.34), (2.35), and Theorem 1.36.

To prove the second assertion, we translate the claimed generators into the setting of tempered sequences. As before (cf. diagram (2.24)), this is done by applying the isomorphism $C^\bullet(\tilde{\mathcal{A}}_\theta) \xrightarrow{\cong} \tilde{C}^\bullet(\tilde{\mathcal{A}}_\theta)$, followed by $\psi_\bullet \circ h^*_\bullet$. This identifies

- τ with $\chi^{(0,0)} \in \mathrm{TS}(\Lambda^0\mathbb{C}^2)$,
- ϕ_j with $\chi^{(0,0)}e_j \in \mathrm{TS}(\Lambda^1\mathbb{C}^2)$, $j = 1,2$, and
- ϕ_{12} with $\chi^{(0,0)}e_1 \wedge e_2 \in \mathrm{TS}(\Lambda^2\mathbb{C}^2)$,

where $\chi^{(0,0)} \in \mathrm{TS}(\mathbb{C})$ is the sequence with entry 1 at $(0,0)$ and 0 everywhere else. For example, ϕ_1 corresponds to $f \in \mathrm{Hom}_{\tilde{\mathcal{A}}_\theta^e}(\tilde{\mathcal{A}}_\theta, \tilde{\mathcal{A}}_\theta^*)$ given by

$$f(1_e \otimes a_1)(a_0) = \tau(a_0\delta_1(a_1)), \quad a_0, a_1 \in \tilde{\mathcal{A}}_\theta,$$

and we have

$$\begin{aligned} h_1^*(f)(1_e \otimes e_1)(U^rV^s) &= f((U^{-1} \otimes 1) \otimes U)(U^rV^s) \\ &= \tau(U^rV^sU^{-1}\delta_1(U)) \\ &= \tau(U^rV^s) = \chi_{r,s}^{(0,0)}, \end{aligned}$$

whereas

$$h_1^*(f)(1_e \otimes e_2)(U^rV^s) = 0.$$

Thus, the proposed generators correspond precisely to a basis of the tempered sequences in $\Lambda^*\mathbb{C}^2$ supported only at $(0,0)$, and as we have seen, these can be identified with $HP(\tilde{\mathcal{A}}_\theta)$.

Lastly, let $n = 0, 1, 2$. The image of $HC^n(\tilde{\mathcal{A}}_\theta)$ in $HH^n(\tilde{\mathcal{A}}_\theta)$ under I then includes the degree n generator(s) of $HP(\tilde{\mathcal{A}}_\theta)$ from part (i). Moreover, since $HC^n(\tilde{\mathcal{A}}_\theta) \subseteq \ker IB$ (by definition of B), the space

$$I(HC^n(\tilde{\mathcal{A}}_\theta))/\operatorname{im}(IB) \cong HC^n(\tilde{\mathcal{A}}_\theta)/(\operatorname{im} B + \ker I)$$

can be identified with a subspace of $H^n(HH^\bullet(\tilde{\mathcal{A}}_\theta), IB)$. This subspace contains the generator(s), so it is precisely the span of the generator(s). It follows that $HC^n(\tilde{\mathcal{A}}_\theta)$ is a direct sum of said span and $\operatorname{im} B + \ker I$. By Theorem 1.30, the latter is the same as $\ker S + \operatorname{im} S$, and the claims of (iii) follow. $\square$

3 Difference-Differential Equations

The aim of this chapter is to introduce an application of the understanding of non-commutative tori built up in chapter 2, in particular of $HC^{ev}(\tilde{\mathcal{A}}_\theta)$. The application in question is to a class of difference-differential equations on the real line. In section 3.1, this is set up by equipping the Schwartz space $\mathcal{S}(\mathbb{R})$ with the structure of a non-commutative line bundle over $\tilde{\mathcal{A}}_\theta$. A particular connection on this bundle is then used in section 3.2 to realise a large class of difference-differential operators as (pseudo)differential operators in that setting. As shown by Connes [6], the classical notion of ellipticity generalises well to these operators, and there is in fact an index theorem for such elliptic operators. Section 3.3 briefly illustrates this in examples. This chapter is based on [6].

3.1 $\mathcal{S}(\mathbb{R})$ as a noncommutative vector bundle

As discussed in Section 1.4, smooth vector bundles over non-commutative tori can be realised as finitely generated projective modules over $\tilde{\mathcal{A}}_\theta$. There is in fact a full classification of these modules, see Theorem III.3.14 in [6] (and references there). Among the simplest of them is the (complex-valued) Schwartz space $\mathcal{S}(\mathbb{R})$ with right $\tilde{\mathcal{A}}_\theta$-module structure defined by

$$(f \cdot U)(t) \coloneqq f(t+\theta), \quad (f \cdot V)(t) \coloneqq e^{2\pi i t} f(t) \quad \forall f \in \mathcal{S}(\mathbb{R}), t \in \mathbb{R}, \tag{3.1}$$

where we have *changed our convention to* $\theta \in (0,1]$.

Remark 3.1. It is rather non-obvious that this module is finitely generated projective, and we shall not prove it here. A proof of a more general statement can be found in [17]. To see what happens in the commutative case, i. e. when $\theta = 1$, consider the space of quasiperiodic functions

$$\Gamma \coloneqq \{f \in C^\infty(\mathbb{C}) \mid f(z+1) = f(z), f(z+i) = e^{2\pi i x} f(z) \quad \forall z = x+iy \in \mathbb{C}\},$$

which becomes a module over $C^\infty(\mathbb{T}^2)$ with the actions

$$(f \cdot U)(x+iy) = e^{2\pi i x} f(x+iy), \quad (f \cdot V)(x+iy) = e^{2\pi i y} f(x+iy).$$

This is precisely the module of sections corresponding to the complex line bundle over $\mathbb{T}^2 \cong \mathbb{C}/\mathbb{Z}^2$ obtained from $\mathbb{C} \times \mathbb{C}$ by factoring out the $\mathbb{Z}$-action

$$(m,n) \cdot (z,\zeta) \coloneqq (z+m+in, e^{2\pi i \operatorname{Re} z}\zeta), \quad m,n \in \mathbb{Z},\ z,\zeta \in \mathbb{C}.$$

As is explained in [7], any element $f \in \Gamma$ can be written uniquely as

$$f(x+iy) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \tilde{f}(y-n), \quad x, y \in \mathbb{R}$$

for a Schwartz function $\tilde{f} \in \mathcal{S}(\mathbb{R})$. This yields a $\mathbb{C}$-linear isomorphism $\Gamma \to \mathcal{S}(\mathbb{R})$, $f \mapsto \tilde{f}$. The induced $C^\infty(\mathbb{T}^2)$-module structure on $\mathcal{S}(\mathbb{R})$ is precisely the one given in (3.1).

We will also need the following explicit description of the endomorphisms of this module [6][4]: one has $\mathrm{End}_{\tilde{\mathcal{A}}_\theta}(\mathcal{S}(\mathbb{R})) \cong \tilde{\mathcal{A}}_{1/\theta} = \langle U', V' : V'U' = e^{2\pi i/\theta} U'V' \rangle$, such that the generators correspond to the maps given by

$$(U'(f))(t) = f(t+1), \quad (V'(f))(t) = e^{-2\pi i t/\theta} f(t), \quad f \in \mathcal{S}(\mathbb{R}), t \in \mathbb{R}. \tag{3.2}$$

Thus, a general module endomorphism of $\mathcal{S}(\mathbb{R})$ is represented by $\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}(V')^k(U')^\ell \in \tilde{\mathcal{A}}_{1/\theta}$, which acts as

$$\mathcal{S}(\mathbb{R}) \ni f \longmapsto \sum_{\ell\in\mathbb{Z}} g_\ell\, f(\cdot + \ell), \tag{3.3}$$

where $g_\ell \in \mathbb{C}^\infty(\mathbb{R})$ is the periodic function of period θ given by the Fourier series $g_\ell(t) = \sum_{k\in\mathbb{Z}} a_{-k,\ell} e^{2\pi k i t/\theta}$.

Let us now construct a connection on the module $\mathcal{S}(\mathbb{R})$ with respect to the dg algebra $\Omega_\theta = \tilde{\mathcal{A}}_\theta \otimes \Lambda^*\mathbb{C}^2$ (defined in (2.18)). The construction can be motivated from the commutative case $\theta = 1$: to fully define a connection ∇ on a vector bundle E over $\mathbb{T}^2$, it is sufficient to define $\nabla(s)(\partial_j)$ for some global frame $\{\partial_1, \partial_2\}$, for every section $s \in \Gamma(E)$. As discussed in Remark 2.13, the derivations δ_1, δ_2 on $\tilde{\mathcal{A}}_\theta$ can be viewed as such a global frame. To put this idea to use, it remains to generalise the action of $\Omega^1(\mathbb{T}^2)$ on vector fields to an action of Ω^1_θ on derivations of $\tilde{\mathcal{A}}_\theta$. For such a derivation δ, and $\omega = a \otimes e_1 + b \otimes e_2 \in \Omega^1_\theta$, one is led to define

$$\omega(\delta) \coloneqq a\delta(U) + b\delta(V) \in \tilde{\mathcal{A}}_\theta. \tag{3.4}$$

Using (3.4) (heuristically), we may decompose a linear map $\nabla\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega^1_\theta$ as

$$\nabla f = \nabla_1(f) \otimes (1 \otimes e_1) + \nabla_2(f) \otimes (1 \otimes e_2) \in \mathcal{S}(\mathbb{R}) \otimes \Omega^1_\theta, \quad f \in \mathcal{S}(\mathbb{R}), \tag{3.5}$$

where $\nabla_j(f) \in \mathcal{S}(\mathbb{R})$ is interpreted as the covariant derivative of f by δ_j. Then one checks that ∇ is a connection if and only if

$$\nabla_j(fa) = \nabla_j(f)a + f\delta_j(a) \quad \forall f \in \mathcal{S}(\mathbb{R}), a \in \tilde{\mathcal{A}}_\theta, j = 1, 2. \tag{3.6}$$

In what follows, we will be using the following choice of $\nabla_1, \nabla_2\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ [6, p. 349], which defines a connection by (3.6):

$$\nabla_1(f)(t) = \frac{-2\pi i t}{\theta} f(t), \quad \nabla_2(f)(t) = f'(t) \quad \forall f \in \mathcal{S}(\mathbb{R}), t \in \mathbb{R}. \tag{3.7}$$

To characterise this connection and the given non-commutative vector bundle, one may compute the curvature form for ∇:

Proposition 3.2. *The curvature form of the connection* $\nabla\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega^1_\theta$ *defined by* (3.7) *and* (3.5) *is* $R = \frac{2\pi i}{\theta} \otimes e_1 \wedge e_2 \in \mathrm{End}_{\Omega_\theta}(\mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega_\theta)$.

Proof. By definition, $R = \hat{\nabla}^2$, where $\hat{\nabla}\colon \mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega_\theta \to \mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega_\theta$ is the extension from Proposition 1.45. Since R raises degree by 2, and Ω_θ has top degree 2, it is sufficient to compute R on $\mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega^0_\theta$. Any element of this space can be written as $f \otimes (1 \otimes 1)$ for $f \in \mathcal{S}(\mathbb{R})$ (where the first 1 is in $\tilde{\mathcal{A}}_\theta$ and the second in $\Lambda^*\mathbb{C}^2$). This has the advantage that $d(1 \otimes v) = 0$ in Ω_θ for any $v \in \Lambda^*\mathbb{C}^2$. We now compute

$$\begin{aligned}
R(f \otimes (1 \otimes 1)) &= \hat{\nabla}(\nabla(f)(1 \otimes 1)) \\
&= \hat{\nabla}(\nabla_1(f) \otimes (1 \otimes e_1) + \nabla_2(f) \otimes (1 \otimes e_2)) \\
&= \nabla(\nabla_1(f))(1 \otimes e_1) + \nabla(\nabla_2(f))(1 \otimes e_2) \\
&= (\nabla_1(\nabla_2(f)) - \nabla_2(\nabla_1(f))) \otimes (1 \otimes e_1 \wedge e_2) \\
&= \tfrac{2\pi i}{\theta} f \otimes (1 \otimes e_1 \wedge e_2).
\end{aligned}$$

As a right Ω_θ module endomorphism of $\mathcal{S}(\mathbb{R}) \otimes_{\tilde{\mathcal{A}}_\theta} \Omega_\theta$, this is just the action of $\frac{2\pi i}{\theta} \otimes e_1 \wedge e_2 \in \Omega^2_\theta$ as claimed. □

From here on, one can use Theorem 1.47 to describe the action of $HC^{ev}(\tilde{\mathcal{A}}_\theta)$ on $\mathcal{S}(\mathbb{R})$, and in particular distinguish it from the free module $\tilde{\mathcal{A}}_\theta$ [6, p. 232][5, p. 139].

3.2 Elliptic operators and index

In the previous section, we introduced $\mathcal{S}(\mathbb{R})$ as a finitely generated projective module over $\tilde{\mathcal{A}}_\theta$, and equipped it with a connection ∇. In constructing this connection, we already used the maps $\nabla_1, \nabla_2\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, which can be interpreted as covariant derivatives with respect to basis vector fields. Indeed, in the commutative case $\theta = 1$, they are precisely the covariant derivatives by coordinate vector fields, obtained from ∇ (cf. Remark 2.13). This motivates the following definitions taken from [6]:

Definition 3.3.

(i) A *differential operator (associated to* ∇*)* is a linear operator $D\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ of the form
$$D = \sum_{j=0}^{n} \sum_{\substack{k,\ell \geq 0 \\ k+\ell=j}} c_{k,\ell} \nabla_1^k \nabla_2^\ell,$$
with $c_{k,\ell} \in \mathrm{End}_{\tilde{\mathcal{A}}_\theta}(\mathcal{S}(\mathbb{R})) \cong \tilde{\mathcal{A}}_{1/\theta}$. Assuming $c_{k,\ell} \neq 0$ for some (k, ℓ) with $k + \ell = n$, we call n the *order* of D.

(ii) The *(principal) symbol* of an order n operator D as in (i) is the map $\sigma\colon S^1 \to \tilde{\mathcal{A}}_{1/\theta}$ defined by
$$\sigma(x, y) := \sum_{k+\ell=n} c_{k,\ell}\, x^k y^\ell \quad \forall (x, y) \in S^1 \subset \mathbb{R}^2.$$

D is *elliptic* if $\sigma(x, y)$ is invertible (in $\tilde{\mathcal{A}}_{1/\theta}$) for all $(x, y) \in S^1$.

Despite their abstract origins, differential operators D as defined above describe rather explicit equations, which can be viewed as generalisations of differential equations on the real line. Indeed, ∇_2 acts exactly by differentiation. In addition to differentiation, the action of ∇_1 allows multiplication by polynomials. Lastly, the $c_{k,\ell}$ act as in (3.3), i. e. by forming finite differences, with smooth periodic coefficients. In particular, one can realise many *difference-differential equations* this way. By a difference-differential equation, we mean a functional equation involving differentiation as well as translation of functions $f \mapsto f(\cdot + \ell)$, $\ell \in \mathbb{Z}$. Explicit examples are discussed in section 3.3.

Ellipticity leads to similar regularity results as in more classical settings. Here we view D as above as an (in general) unbounded operator on $L^2(\mathbb{R})$ with domain $\mathcal{S}(\mathbb{R})$. We then have:

Theorem 3.4 ([6] Theorem IV.6.2)**.** *Let D be an elliptic operator as in Definition 3.3. Consider D as a (possibly unbounded) closed operator on $L^2(\mathbb{R})$ (taking the closure if necessary). Then:*

(i) D is Fredholm, and

(ii) for $f \in L^2(\mathbb{R})$ such that $Df = 0$ we must have $f \in \mathcal{S}(\mathbb{R})$.

Moreover, there is a rather explicit index theorem, which relates the Fredholm index $\operatorname{ind} D$ to the symbol and the cyclic cohomology of the locally convex algebra $C^\infty(S^1, \tilde{\mathcal{A}}_{1/\theta}) := C^\infty(S^1) \otimes \tilde{\mathcal{A}}_{1/\theta}$. The formula involves two particular cyclic cocycles of $C^\infty(S^1, \tilde{\mathcal{A}}_{1/\theta})$, which can be constructed as cup products.

To begin with, let $\rho \in HC^1(C^\infty(S^1))$ denote the standard cyclic cocycle obtained from integration as in (1.3). That is,

$$\rho(f_0, f_1) := \int_{S^1} f_0 df_1.$$

Via the cup product, we can immediately define two cyclic cocycles $\tau_1 := \rho \# \tau \in HC^1(C^\infty(S^1, \tilde{\mathcal{A}}_{1/\theta}))$ and $\tau_3 := \rho \# \phi_{12} \in HC^3(C^\infty(S^1, \tilde{\mathcal{A}}_{1/\theta}))$.

Observing that for an elliptic operator in our current sense, the symbol σ and its pointwise inverse σ^{-1} are both elements of $C^\infty(S^1, \tilde{\mathcal{A}}_{1/\theta})$, we can now state the index theorem:

Theorem 3.5 ([6], Theorem IV.6.3)**.** *For an elliptic operator D as in Definition 3.3, we have*

$$\operatorname{ind} D = \frac{1}{6(2\pi i)^2 \theta} \tau_3(\sigma^{-1}, \sigma, \sigma^{-1}, \sigma) - \frac{1}{2\pi i} \tau_1(\sigma^{-1}, \sigma). \tag{3.8}$$

In order to apply this theorem in examples, one needs to explicitly compute the expression on the right-hand side of (3.8). To that end, we unravel the definitions of τ_1 and τ_3. First of all, it is sufficient (by density and multilinearity) to describe τ_1 and τ_3 on tensor products $f \otimes a$, with $f \in C^\infty(S^1)$ and $a \in \tilde{\mathcal{A}}_{1/\theta}$. Secondly, our definition of the cup product (above Proposition 1.29) was in terms of the universal dg algebras, whereas here it would be convenient to use the given dg algebras $\Omega^*(S^1)$ and $\Omega_{1/\theta}$ instead. Using the universal property of the universal dg algebras (see Remark 1.24), one checks that this is indeed possible. That is, τ_1 and τ_3 are the characters of the closed graded traces on $\Omega^*(S^1) \otimes \Omega_{1/\theta}$ defined by $\int_{S^1} \otimes \int_0$ and $\int_{S^1} \otimes \int_{12}$, respectively. (Here $\int_0$, $\int_{12}$ are as defined in Lemma 2.17.) To simplify the notation, we will suppress wedge product signs as well as the inclusion map $\tilde{\mathcal{A}}_{1/\theta} \to \Omega_{1/\theta}$.

Now given $f_0, f_1 \in C^\infty(S^1)$ and $a_0, a_1 \in \tilde{\mathcal{A}}_{1/\theta}$, we have

$$(f_0 \otimes a_0)d(f_1 \otimes a_1) = f_0 df_1 \otimes a_0 a_1 + f_0 f_1 \otimes a_0 da_1.$$

Since $\int_0$ is zero-dimensional, we only need the first summand, finding

$$\tau_1(f_0 \otimes a_0, f_1 \otimes a_1) = \left(\int_{S^1} \otimes \int_0\right)(f_0 df_1 \otimes a_0 a_1) = \int_{S^1} (f_0 df_1)\, \tau(a_0 a_1). \tag{3.9}$$

Similarly, given $f_0, f_1, f_2, f_3 \in C^\infty(S^1)$ and $a_0, a_1, a_2, a_3 \in \tilde{\mathcal{A}}_{1/\theta}$, we have (with the sign rule (1.15))

$$\begin{aligned}
&(f_0 \otimes a_0)d(f_1 \otimes a_1)d(f_2 \otimes a_2)d(f_3 \otimes a_3) \\
&= (f_0 \otimes a_0)(df_1 f_2 f_3 \otimes a_1 da_2 da_3 - f_1 df_2 f_3 \otimes da_1 a_2 da_3 + f_1 f_2 df_3 \otimes da_1 da_2 a_3) \\
&= f_0 df_1 f_2 f_3 \otimes a_0 a_1 da_2 da_3 - f_0 f_1 df_2 f_3 \otimes a_0 da_1 a_2 da_3 \\
&\quad + f_0 f_1 f_2 df_3 \otimes a_0 da_1 da_2 a_3.
\end{aligned}$$

This leads to the formula

$$\begin{aligned}
&\tau_3(f_0 \otimes a_0, f_1 \otimes a_1, f_2 \otimes a_2, f_3 \otimes a_3) \\
&= \int_{S^1} (f_0 df_1 f_2 f_3) \tau\big(a_0 a_1 \delta_1(a_1)\delta_2(a_3) - a_0 a_1 \delta_2(a_1)\delta_1(a_3)\big) \\
&- \int_{S^1} (f_0 f_1 df_2 f_3) \tau\big(a_0 \delta_1(a_1)\delta_2(a_2)a_3 - a_0 \delta_2(a_1)\delta_1(a_2)a_3\big) \\
&+ \int_{S^1} (f_0 f_1 f_2 df_3) \tau\big(a_0 \delta_1(a_1) a_2 \delta_2(a_3) - a_0 \delta_2(a_1) a_2 \delta_1(a_3)\big)
\end{aligned} \tag{3.10}$$

While there are more concise or conceptual ways to express these results (see pages 359 and 360 of [6]), (3.9) and (3.10) are sufficient for simple examples.

3.3 Examples

Here we explore some of the possible applications of Theorem 3.5, starting with simple examples. To begin with, an order 0 operator $D = c_{0,0}$ is simply the action of the element $c_{0,0} = \sum_{r,s\in\mathbb{Z}} (U')^r (V')^s \in \tilde{\mathcal{A}}_{1/\theta}$, and the symbol is the constant map with value $c_{0,0}$. Thus D is elliptic if and only if $c_{0,0}$ is an invertible element of $\tilde{\mathcal{A}}_{1/\theta}$. Hence an order 0 elliptic operator will be invertible and therefore trivially have index zero.

Things already become much more interesting for first order operators, of the general form $D = c_{0,0} + c_{1,0}\nabla_1 + c_{0,1}\nabla_2$. We can write the symbol as

$$\sigma(t) := \sigma(\cos t, \sin t) = \cos t\, c_{1,0} + \sin t\, c_{0,1}, \quad t \in \mathbb{R}.$$

Clearly, D cannot be elliptic if either of $c_{1,0}$, $c_{0,1}$ is zero, or if $c_{1,0}$ is a real scalar multiple $c_{0,1}$. However, other scalar multiples work: for instance $c_{1,0} = U'$ and $c_{0,1} = \pm iU'$ leads to the invertible symbol $\sigma(t) = e^{\pm it}U'$ and, assuming there are no order zero terms, corresponds to the ODE

$$\frac{-2\pi i(t+1)}{\theta} f(t+1) \pm i f'(t+1) = 0, \quad t \in \mathbb{R}$$

which can of course be simplified to

$$f'(t) = \pm\frac{2\pi}{\theta} t f(t), \quad t \in \mathbb{R}. \tag{3.11}$$

Let us compute the index for this operator[1]. Firstly, $\sigma(t)^{-1} = e^{\mp it}(U')^{-1}$. By formula (3.9) we have

$$\tau_1(\sigma^{-1}, \sigma) = \int_0^{2\pi} \pm i\, \tau(1) = \pm 2\pi i,$$

whereas (3.10) leads to

$$\tau_3(\sigma^{-1}, \sigma, \sigma^{-1}, \sigma) = 0 - 0 + 0 = 0,$$

because each summand contains a term $\delta_2((U')^{\pm 1}) = 0$. Altogether, the index formula (3.8) yields $\operatorname{ind} D = \mp 1$.

To interpret this, note that the nonzero solutions of (3.11) are given by $t \mapsto ce^{\pm\pi t^2/\theta}$, $c \in \mathbb{C}$. For the equation with a positive sign, this means that there is no nontrivial square integrable solution, i. e. the kernel is trivial and hence the co-kernel is one-dimensional. In the version with a negative sign, we see that the kernel is one-dimensional, so the cokernel must be trivial. In other words, the inhomogeneous equation

$$f'(t) + \frac{2\pi}{\theta} t f(t) = g(t), \quad t \in \mathbb{R}$$

has a solution for every $g \in L^2(\mathbb{R})$. Note also that the existence of a nontrivial solution was guaranteed by the positive index in this case.

For the ODE (3.12), the above results are of course not hard to verify directly. However, we are free to include an order zero term $c_{0,0}$, turning (3.11) into a difference-differential equation

$$f'(t) = \frac{2\pi}{\theta} t f(t) + \sum_{\ell \in \mathbb{Z}} g_\ell(t) f(t+\ell), \quad t \in \mathbb{R} \tag{3.12}$$

[1] these two operators to be precise, one for each sign choice

as in (3.3). Even for simple choices of the g_ℓ (e. g. finitely many constant functions), the resulting equations become significantly harder to analyse, but the corresponding operator remains elliptic with unchanged index.

It should be noted that the constant $2\pi/\theta$ in (3.11) is of no importance in this example. Indeed, taking $c_{0,1} = aiU'$ for $a \in \mathbb{R} \setminus \{0\}$ still yields an elliptic operator, and an only slightly more complicated computation shows that its index is $\operatorname{sgn}(a)$. Moreover, replacing U' with V' in the above leads to the same equation.

In fact, the previous example does not make much use at all of the noncommutative setting (hence the minor role of θ), as it involves only one generator of $\tilde{\mathcal{A}}_{1/\theta}$. Perhaps the simplest attempt for an operator that combines U' and V' would be $c_{1,0} = \alpha U'$, $c_{0,1} = \beta V'$ for some $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. However, for $t \notin \frac{\pi}{2} + \pi\mathbb{Z}$, $\sigma(t)$ is invertible if and only if $-\beta/\alpha \tan(t)$ is not in the spectrum of $(U')(V')^{-1}$. A simple computation shows that this spectrum is S^1, so independent of α, β there will always be some t for which invertibility fails. To obtain an elliptic operator one has to work harder.

In the previous paragraph, the issue was that the (real) line through the origin of direction $-\beta/\alpha$ in $\mathbb{C}$ has to intersect the spectrum of $(U')(V')^{-1}$. Replacing $(U')(V')^{-1}$ by a general element $a \in \tilde{\mathcal{A}}_{1/\theta}$, one can avoid this problem by choosing a with suitable spectrum. In particular there is no problem when the spectrum of a lies on a ray not intersecting the origin, i. e. when a is a suitable scalar multiple of a strictly positive element. This is in fact already reflected in our first example, corresponding to a being a non-zero scalar. Elliptic operators arising in this way from strictly positive elements of $\tilde{\mathcal{A}}_{1/\theta}$ are discussed in [6, p. 354].

To illustrate the role that the parameter θ can play for the index, let us mention another example from [6]. Here we assume $\theta \notin \mathbb{Q}$, in which case $\tilde{\mathcal{A}}_{1/\theta}$ contains a nontrivial self-adjoint idempotent known as the[2] *Powers-Rieffel idempotent* p such that $\tau(p) = 1/\theta$. [16] To construct it, one realises $\mathcal{A}_{1/\theta}$ as a C*-subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ in the same way as in (3.2). As in (3.3), this lets us write elements of $\tilde{\mathcal{A}}_{1/\theta}$ as $\sum_{\ell\in\mathbb{Z}} g_\ell (U')^\ell$, where g_ℓ denotes the multiplication operator associated to a smooth θ-periodic function g_ℓ. One makes the ansatz $p = f_{-1}(U')^{-1} + f_0 + f_1 U'$ for θ-periodic smooth functions f_{-1}, f_0, f_1. Self-adjointness of p then necessitates that $f_0 = \overline{f_0}$ and $f_{-1} = f_1 = \overline{f_1}(\cdot - 1)$. Furthermore, idempotence leads to the further conditions

- $f_1 f_1(\cdot + 1) = 0$,
- $(f_0 + f_0(\cdot + 1))f_1 = f_1$, and
- $f_0^2 + |f_1|^2 = f_0$.

These conditions can be fulfilled by choosing f_0 and f_1 accordingly. The details can be found in the original paper [16] or also in [6], which also contains the following result:

Proposition 3.6 ([6], Corollary IV.6.5)**.** *Let g_{-1}, g_0, g_1 be smooth θ-periodic functions*

[2]There isn't a unique such idempotent, but there is a standard construction for such idempotents.

such that

$$\sum_{\ell=-1}^{1} |f_\ell(t) - g_\ell(t)| \leq 1 \quad \forall t \in \mathbb{R} \tag{3.13}$$

for f_{-1}, f_0, f_1 *from a fixed Powers-Rieffel idempotent. Then the difference-differential operator*

$$(Du)(t) := tu(t) - \sum_{\ell=-1}^{1} g_\ell(t) u'(t+\ell), \quad u \in \mathcal{S}(\mathbb{R}) \tag{3.14}$$

is elliptic with index $1 + 2\lfloor 1/\theta \rfloor$.

Here, the index visibly depends on θ: the smaller the parameter, the larger the index. Changing θ also changes the period of the g_ℓ, which can of course be avoided by rescaling all functions involved, e. g. replacing f_ℓ by $f_\ell(\theta\cdot)$. In this setting, we must have $f_1 f_1(\cdot + \theta) = 0$, which results in a shrinking of $\operatorname{supp}(f_1)$ as θ approaches 0 (i. e. as the index grows), so the $g_{\pm 1}$ become progressively more localised. Under this point of view, (3.13) is a family of conditions, parameterised by θ, to be placed on a difference-differential operator of the form (3.14). Loosely speaking, a more extreme condition (small θ) corresponds to a higher index.

Glossary: Special Notation

$(C^\bullet(A), b)$ Hochschild cochain complex over A. 7

$(C^\bullet_\lambda(A), b)$ cyclic cochain complex over A. 11

ΩA universal dg algebra over the algebra A. 12

$\hat{c}$ closed graded trace representing c. 14

I inclusion operator $HC^\bullet_\lambda(A) \to HH^\bullet(A)$. 15

B Connes operator $HH^\bullet_\lambda(A) \to HC^\bullet(A)$. 15

$\#$ cup product. 18

S periodicity operator $HC^\bullet_\lambda(A) \to HC^\bullet(A)$. 19

$\mathcal{A}_\theta$ rotation algebra with parameter θ. 35

q defined by $q \coloneqq e^{2\pi i\theta}$. 35

$\tilde{\mathcal{A}}_\theta$ dense subalgebra of $\mathcal{A}_\theta$ interpreted as smooth functions. 35

τ faithful trace on $\mathcal{A}_\theta$. 37

δ_j basic derivations on $\tilde{\mathcal{A}}_\theta$, $j = 1, 2$. 42

ϕ_j distinguished cyclic cocycles on $\tilde{\mathcal{A}}_\theta$, $j = 1, 2, 12$. 43

$\sum_{k=0}^{n}{}'$ modified sum notation. 45

$\mathrm{TS}(X)$ tempered sequences valued in X. 47

T_v wedge product with v. 52

References

[1] José M. Gracia Bondía, Joseph C. Varilly, and Héctor Figueroa. *Elements of Noncommutative Geometry*. Birkhäuser, 2001.

[2] Marc De Brabanter. “The classification of rational rotation C*-algebras”. In: *Archiv der Mathematik* 43 (1984), pp. 79–83.

[3] Jacek Brodzki. *An Introduction to K-theory and Cyclic Cohomology*. 1996. URL: https://arxiv.org/abs/funct-an/9606001.

[4] Alain Connes. “C* algèbres et géométrie differentielle”. In: *C.R. Acad. Sci. Paris Sér. A-B* 290.13 (1980), A599–A604.

[5] Alain Connes. “Non-commutative differential geometry”. In: *Publications mathématiques de l'I.H.É.S.* 62 (1985), pp. 41–144.

[6] Alain Connes. *Noncommutative Geometry*. Academic Press, 1994.

[7] Francesco D'Andrea, Gaetano Fiore, and Davide Franco. “Modules Over the Noncommutative Torus and Elliptic Curves”. In: *Letters in Mathematical Physics* 104.11 (2014), pp. 1425–1443.

[8] Kenneth R. Davidson. *C*-Algebras by Example*. American Mathematical Society, 1996.

[9] Masoud Khalkhali. *Basic Noncommutative Geometry*. European Mathematical Society, 2013.

[10] Wayne M Lawton. *Tutorial on Rational Rotation C^*–Algebras*. 2021. URL: https://arxiv.org/abs/2111.02932.

[11] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 2001.

[12] John W. Milnor and James D. Stasheff. *Characteristic Classes*. Princeton University Press, 1974.

[13] Ryszard Nest. “Cyclic Cohomology of Non-commutative Tori”. In: *Can. J. Math.* XL.5 (1988), pp. 1046–1057.

[14] Jet Nestruev. *Smooth Manifolds and Observables*. Springer-Verlag, 2003.

[15] Georges de Rham. *Differentiable Manifolds*. Springer-Verlag, 1984.

[16] Marc A. Rieffel. “C*-Algebras Associated with Irrational Rotations”. In: *Pacific J. Math.* 93.2 (1981), pp. 415–429.

[17] Marc A. Rieffel. “Projective Modules over Higher-Dimensional Non-commutative Tori”. In: *Can. J. Math.* XL.2 (1988), pp. 257–338.

[18] Richard G. Swan. "Vector bundles and projective modules". In: *Trans. Amer. Math. Soc.* 105 (1962), pp. 264–277.